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THE PENETRATION OF ATOMIC PARTICLES THROUGH MATTER

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INTRODUCTION

The phenomenon of the scattering and stopping of high speed atomic particles in passing through matter and the accompanying ionization and radiation effects have, as is well known, been one of the most important sources of information regarding the constitution of atoms. Ever since the pioneering work of THOMSON and RUTHERFORD, the analysis of the penetration phenomena has been in continual progress and has, in particular, offered many important tests of the gradually refined methods of atomic mechanics. In the course of this development, the topic has been much discussed within the group working at the Institute for Theoretical Physics in Copenhagen and, in this connection, commemoration is above all due the stimulation of E. J. WILLIAMS, whose premature death has been so deplorable a loss. Already about ten years ago, plans were laid for a general treatment of the problem by WILLIAMS and the writer but, due to the isolation brought about by the war, these plans eventually had to be abandoned.

In recent years, the subject has acquired renewed interest by the discovery that heavy atomic nuclei, in the so-called fission process, may break up into two fragments of comparable mass and charge ejected with kinetic energies of the order of 100 MeV. This phenomenon has provided the possibility of studying the penetration through matter of high speed particles with masses and charges many times larger than those of the particles previously accessible to such investigations. Because of these properties of the fission fragments, several features of only minor consequence in the behaviour of lighter particles are here of determining influence on the whole phenomenon. On these problems a number of experimental and theoretical investigations have been published and, as previously announced, a more comprehensive treatment was prepared by the writer already in 1942. The postponement of the publication, due to circumstances, has, however, offered the opportunity of taking into account important results of more recent researches in this field.

Following the original plan, the subject is treated in a broad manner and emphasis is laid on points illustrative of the more general principles especially regarding the scope of the methods applied. For this reason, many mathematical details which have been thoroughly investigated by other workers have been referred to in a somewhat cursory way and reference to various points of less direct relation to the main subject has been printed in smaller type. At several places, only a qualitative treatment has been given which obviously may require further elaboration. As regards many of the problems, readers may find fuller information in the admirable account of Bethe and Livingston (1937) and also in a forthcoming monograph by Dr. R. L. Platzman, who has kindly shown me those instalments which have already been completed.

At the conclusion of the present work, the writer wishes to acknowledge his debt to many of the present and earlier collaborators at the Institute for numerous illuminating discussions. His thanks are especially due Dr. Stefan Rozental, M.Sc. Aage Bohr and M.Sc. Børge Madsen for valuable help in preparing the manuscript and the illustrating figures.

CHAPTER 1

Scattering of Charged Particles in Atomic Fields.

§ 1.1. Coulomb Interaction in Classical Mechanics.

For the treatment of the interaction of an atomic particle with the matter through which it penetrates, the problem of the collision between two point charges is of primary importance. Although a closer consideration shows that many penetration phenomena depend essentially on the forces acting between the individual constituents of the atoms and may even be influenced by the interaction of neighbouring atoms in the stopping material, we shall, therefore, first consider the simple Coulomb interaction in some detail. In this connection, we shall in the present paragraph recall the well-known treatment of the problem by classical methods and, apart from references at certain points to modifications brought about by relativity theory, we shall, for simplicity, in general assume the relative velocities of the particles to be small compared with the velocity of light. In following paragraphs, we shall proceed to the implications of quantum mechanics and shall especially discuss the limitations imposed on the application of orbital pictures.

In ordinary mechanics, the problem of a collision between two particles attracting or repelling each other with a force inversely proportional to the square of the distance has a particularly simple solution. In the system of reference where the centre of gravity is at rest, the particles will, as is well known, move in hyperbolic orbits with this centre as common focus. By introducing relative coordinates, the problem is even simplified to that of the motion of a particle with the so-called reduced mass

$$m_0 = \frac{m_1 m_2}{m_1 + m_2} \tag{1.1.1}$$

in a fixed radial field with a potential

$$P(r) = \frac{e_1 e_2}{r}. \tag{1.1.2}$$

In these expressions m_1, m_2 and e_1, e_2 denote the masses and charges of the colliding particles, while r represents their distance apart.

Except for scale, the relative orbit is similar to the orbital motions of the particles round the centre of gravity and, denoting the angle between the asymptotes of the hyperbola, the so-called relative deflection angle, by ϑ, we get by a straightforward calculation (cf., for instance, Thomson 1906, p. 376)

$$\operatorname{tg} \frac{\vartheta}{2} = \frac{b}{2p}, \tag{1.1.3}$$

where p is the "impact parameter", defined as the distance at which the particles would pass each other if no forces acted between them, and

$$b = \frac{2|e_1 e_2|}{m_0 v^2} \tag{1.1.4}$$

is a length depending on the relative velocity v and which, in case of repelling particles, just represents the minimum distance of approach in a head-on collision. For attractive as well as for repulsive forces, $\vartheta \geqq \frac{\pi}{2}$ corresponds to $p \leqq \frac{b}{2}$ and the cross-section for backward scattering in the relative motion is, therefore, $\frac{\pi}{4}\, b^2$. For this reason, b will in the following be referred to as the "collision diameter".

From the motion in the centre of gravity system, the actual velocities of the particles can be simply obtained by superposing the uniform velocity v_c of the mass centre. In ordinary penetration phenomena, where the velocity of the incident particle, in the following referred to as particle 1, is very large compared with the thermal velocities of the atoms in the matter, the particle which is hit (particle 2) may often be considered initially at rest, and we have, therefore,

$$v_c = v \cdot \frac{m_1}{m_1 + m_2}. \tag{1.1.5}$$

For this case, Fig. 1 illustrates the determination of the velocities v_1 and v_2 of the particles after a collision of relative deflection angle ϑ.

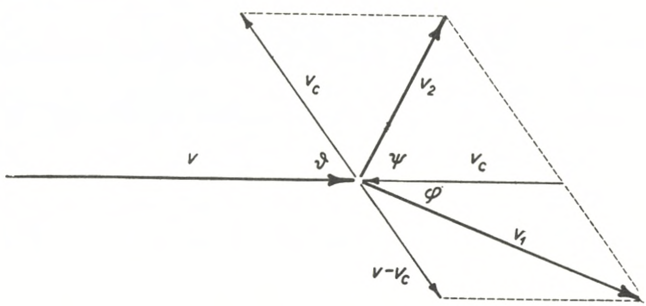

Fig. 1.

In the applications, we are most directly concerned with the deflection angles φ and ψ of the particles and with the energy transfer in the collision. From the figure it is seen, by means of (1.1.5), that

$$\operatorname{tg}\varphi = \frac{(v - v_c)\sin\vartheta}{v_c + (v - v_c)\cos\vartheta} = \frac{m_2 \sin\vartheta}{m_1 + m_2\cos\vartheta}; \qquad (1.1.6)$$

while for ψ we get simply

$$\psi = \frac{\pi}{2} - \frac{\vartheta}{2}. \qquad (1.1.7)$$

When ϑ varies from 0 to π, the angle ψ will thus decrease from $\frac{\pi}{2}$ to 0. The relationship between φ and ϑ, however, will depend on the mass-ratio. If $m_1 < m_2$, the angle φ will increase steadily with ϑ from 0 to π, while for $m_1 > m_2$ it is seen that φ passes through a maximum value smaller than $\frac{\pi}{2}$. In the special case where the masses are equal, we have simply $\varphi = \frac{\vartheta}{2}$.

Since, further, $v_2 = 2\, v_c \sin\frac{\vartheta}{2}$, we get from (1.1.5) for the energy T transferred to particle 2 during the collision

$$T = \frac{1}{2} m_2 v_2^2 = \frac{2\, m_1^2 m_2}{(m_1 + m_2)^2} v^2 \sin^2\frac{\vartheta}{2} = T_m \sin^2\frac{\vartheta}{2}, \qquad (1.1.8)$$

where T_m represents the maximum energy transfer,

$$T_m = 2\frac{m_0^2}{m_2} v^2 = \frac{4\, m_1 m_2}{(m_1 + m_2)^2} E, \qquad (1.1.9)$$

m_0 being given by (1.1.1) and E representing the kinetic energy, $E = \frac{1}{2} m_1 v^2$, of the incident particle. In case $m_1 = m_2$, we have, of course, $T_m = E$, while $T_m \ll E$ if m_1 is either very large or very small compared with m_2. Introducing for ϑ the formula (1.1.3), one finds, by means of (1.1.4),

$$T = \frac{2\, e_1^2 e_2^2}{m_2 v^2} \frac{1}{p^2 + \frac{b^2}{4}} \tag{1.1.10}$$

as expressed in terms of the impact parameter.

In distant collisions where $p \gg b$ and where, consequently, according to (1.1.3), the relative deflection angle is very small, and approximately equal to $\frac{b}{p}$, the two-body problem becomes especially simple. In particular, as is seen from (1.1.7), the struck particle will be set in motion practically perpendicular to the direction of the incident particle. This result also immediately follows from a consideration of the force exerted on particle 2 during a distant collision, in which the motion of particle 1 may be regarded as approximately undisturbed. It is true that the displacement of the struck particle parallel to the direction of the incident particle would become infinite for a pure two-body collision, but such displacements are seen, for symmetry reasons, not to affect the resultant momentum and energy transfer.

For the total momentum transfer, M, we thus get for distant collisions, by neglecting the displacement of the struck particle during the encounter[1],

$$M \approx \int_{-\infty}^{\infty} \frac{|e_1 e_2|\, p}{(p^2 + v^2 t^2)^{3/2}}\, dt = \frac{2\,|e_1 e_2|}{p v}, \tag{1.1.11}$$

from which follows

$$T = \frac{M^2}{2\, m_2} \approx \frac{2\, e_1^2 e_2^2}{m_2 v^2} \frac{1}{p^2} \tag{1.1.12}$$

1) The notation $\approx$ is used when two quantities are asymptotically equal, while two quantities which are merely of the same order of magnitude will be connected by the symbol $\sim$.

just corresponding to formula (1.1.10) for $p \gg b$ or $T \ll T_m$.

The displacement q of the struck particle in the direction perpendicular to the path of the incident particle may, as regards order of magnitude, be simply estimated from the momentum transfer and the effective duration of the collision. Since the main part of the interaction takes place within a time interval of about $2p/v$, we find from (1.1.4) and (1.1.11) for this displacement

$$q \sim \frac{M}{m_2}\frac{2p}{v} \approx \frac{2\,|\,e_1 e_2\,|}{m_2 v^2} = b\,\frac{m_0}{m_2}, \qquad (1.1.13)$$

an estimate which shows that, in distant collisions, q is small compared with p and, to a first approximation, independent of the impact parameter.

At the same time it is, of course, the small variation, due to this displacement, of the forces exerted by particle 2 on particle 1 which is the origin of the loss of kinetic energy of the incident particle during the collision. In fact, the change in the component of this force directed against the motion of the particle is approximately $\frac{e_1 e_2}{p^3}\,q$ and, acting through a distance comparable with p, it causes an energy loss just corresponding to (1.1.12). In the following chapters, we shall make illustrative use of these simple considerations.

In relativity theory, a rigorous treatment of the collisions between charged particles presents in general a highly complicated problem. In the particular case where one of the particles is very much heavier than the other, the problem has been treated with neglect of radiative forces by Darwin (1913), who showed that for impact parameters smaller than a certain critical value which, for velocities about half that of light is approximately equal to b, a collision between charges of opposite sign should even result in a coalescence of the charges. Apart from such aspects of the problem, which appear essentially different in quantum theory, it is, of course, necessary also to take into account that the relationships between energy and momentum transfer and angular deflections may in relativistic mechanics deviate considerably from those given above. A survey of such relationships to be applied in the examination of tracks of high speed particles has recently been given by BLATON (1948).

In distant collisions where the velocity of the struck particle remains small compared with the light velocity c, it can be easily shown that the asymptotic expressions (1.1.11) and (1.1.12) for M and T will be valid also in the relativistic case. In fact, the field of the incident particle will simply be contracted in the direction of its motion in the ratio $\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}$ and, as regards the transfer of momentum, the ensuing shortening of the duration of the impact will be just compensated by a corresponding increase in the intensity of the force during the collision. On the other hand, the displacement q of the struck particle may be considerably smaller than given by (1.1.13). An analysis of the energy balance is in this case somewhat more intricate, since it is necessary to take into account the retardation of the reactive forces (cf. A. BOHR 1948).

§ 1.2. Statistics of Collisions. Rutherford Scattering Law.

In classical mechanics, the differential cross-section $d\sigma$ for collisions with impact parameters between p and $p + dp$ is, of course, $2\pi p dp$. Introducing, by means of (1.1.3), the deflection angle in relative motion, we thus get

$$d\sigma = \frac{\pi b^2}{4} \cos\frac{\vartheta}{2} \operatorname{cosec}^3\frac{\vartheta}{2}\, d\vartheta. \qquad (1.2.1)$$

Since the corresponding solid angle is given by $d\omega = 2\pi \sin\vartheta\, d\vartheta$, we have, from (1.1.4),

$$d\sigma = R(\vartheta)\, d\omega = \left(\frac{e_1 e_2}{2\, m_0 v^2}\right)^2 \operatorname{cosec}^4\frac{\vartheta}{2}\, d\omega, \qquad (1.2.2)$$

an expression well known from RUTHERFORD's fundamental researches (1911) on the scattering of α-rays.

Formula (1.2.2) can be directly applied to the large angle scattering of α-rays by heavy nuclei, where ϑ is approximately equal to the actual deflection angle φ but, when considering particles of comparable masses, it is necessary, for comparison with experiments, to introduce φ by means of expression (1.1.6), which gives in general a somewhat more complicated scattering law (DARWIN 1914). In distant collisions, however, where the angle ϑ is small, (1.2.1) reduces to the simple expression

$$d\sigma \approx 2\pi \left(\frac{2\, e_1 e_2}{m_0 v^2}\right)^2 \frac{d\vartheta}{\vartheta^3} \tag{1.2.3}$$

and, since (1.1.6) gives $\varphi \approx \frac{m_2}{m_1 + m_2}\, \vartheta$, we have

$$d\sigma \approx 2\pi \left(\frac{2\, e_1 e_2}{m_1 v^2}\right)^2 \frac{d\varphi}{\varphi^3} \tag{1.2.4}$$

by means of (1.1.1).

For velocities approaching that of light, formula (1.2.2) requires modifications especially for large angles while, on the lines of the argumentation in § 1.1, it is seen that, in distant collisions, relativity corrections simply amount to a replacement in (1.2.4) of the rest mass m_1 by the effective mass $m_1 \gamma$.

In quantum mechanics, the whole idea of orbits and, particularly, of impact parameters has only a restricted validity, but nevertheless the notion of cross-section may be conveniently applied. Even if it cannot be simply pictured as a target area, it may be defined, in an equivalent manner, by the number of collisions with specified results taking place per unit time, divided by the current density of the incident beam of particles. In general, of course, cross-sections deduced in quantum mechanics may have essentially different values from those obtained from classical mechanical calculations, but, in the special case of the scattering by a fixed Coulomb field, it is a well-known result of the wave-mechanical analysis that, apart from relativity modifications, formula (1.2.2) holds quite generally for the statistical distribution of the relative deflection angles (GORDON 1928).

It may be noted that, in quantum mechanics, we meet with special features when considering a collision between identical particles where peculiar exchange phenomena occur, connected with the impossibility of distinguishing between the individual particles during their interaction. As shown by MOTT (1930), the Rutherford law must in such cases be replaced by the expression

$$\left.\begin{aligned} d\sigma = \left(\frac{e^2}{mv^2}\right)^2 \Big(\operatorname{cosec}^4 \frac{\vartheta}{2} + \sec^4 \frac{\vartheta}{2} + \\ \begin{bmatrix} +2 \\ -1 \end{bmatrix} \operatorname{cosec}^2 \frac{\vartheta}{2} \sec^2 \frac{\vartheta}{2} \cos\left\{\frac{2e^2}{\hbar v} \log \operatorname{tg} \frac{\vartheta}{2}\right\}\Big)\, d\omega \end{aligned}\right\} \quad (1.2.5)$$

where e and m are the charge and mass of the particles, and where $\hbar$ is PLANCK's constant divided by 2π. The upper and lower factors in the square brackets refer to particles of spin 0 and $\frac{1}{2}$ obeying Bose-Einstein and Fermi-Dirac statistics, respectively. While the two first terms in (1.2.5) simply correspond to the probability, according to the Rutherford law, of either particle being scattered into the specified angular region $d\omega$, the third term represents the exchange effect, which is a purely quantum-mechanical phenomenon.

Due to the circumstance that, disregarding relativity refinements and specific exchange effects, the statistical laws governing two-body collisions are the same in quantum mechanics as in classical mechanics, several results in penetration theory, obtained by classical mechanical methods, are valid far beyond the scope of orbital pictures. Essential features of the scattering and stopping problems, however, are determined by the fact that the matter penetrated does not consist of free particles, but of atoms containing electrons bound by nuclei. The interaction between the penetrating particle and the matter cannot, therefore, be accounted for by simple two-body collisions. Not only will there be a partial compensation of the forces exerted on the incident particle but, also, the atomic binding forces may influence the course of the collisions. In many problems it is, therefore, necessary to take into account certain screening effects, of static or dynamic character, the influence of which may be essentially different in classical mechanics and in quantum theory.

In the subsequent paragraphs of this chapter, we shall, therefore, attempt a comprehensive treatment of the various aspects of the screening problems. We meet here with a number of paradoxes, the elucidation of which offers illustrative examples of the application of the ideas of indeterminacy and complementarity in quantum theory. These problems have been the subject of numerous discussions referred to in the Introduction and have been interestingly expounded by various authors (Bloch 1933, Williams 1933 and, especially, Williams 1945).

§ 1.3. Criterion for Application of Orbital Pictures in Coulomb Scattering.

In order to make the trend of the argument as clear as possible, we shall begin by examining the conditions for the unambiguous use of orbital pictures in the simple case of the scattering of charged particles in a fixed Coulomb field.

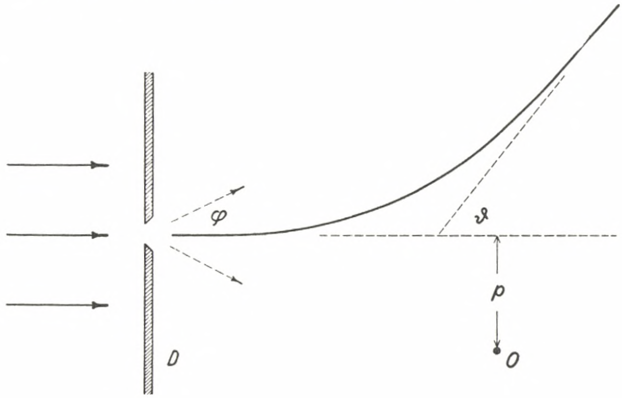

Fig. 2.

In Fig. 2, let O represent the force centre and the full-drawn curve the hyperbolic orbit to be expected from classical mechanics for a particle with impact parameter p. By D is indicated a suitably placed, fixed diaphragm with a hole serving to secure an actual localization of the path of the particle.

As is well known, the presence of such a diaphragm will, according to quantum mechanics, cause a diffraction indicated by dotted arrows in the figure, and the problem is now whether this diffraction will be small compared with the deflection of the particle due to the field or whether it will be so large that it will completely overshadow any orbital deflection. In the former case, it is in fact possible to construct wave packets which, to a high degree of approximation, follow the classical orbits while, in the latter case, we must be prepared for proper quantum effects which evade any analysis in terms of ordinary mechanical pictures.

For a circular hole of diameter d, the aperture of diffraction will, as regards order of magnitude, be given by

$$\delta\varphi = \frac{\lambda\!\!\!^{-}}{d}, \qquad (1.3.1)$$

where

$$\lambda\!\!\!^{-} = \frac{\hbar}{m_0 v} \qquad (1.3.2)$$

is the de Broglie wave-length divided by 2π. For a sharply limited hole, the problem of diffraction is somewhat complicated but, if we assume that the hole is limited by partly permeable edges in such a way that the intensity of the penetrating beam at different distances from the centre of the hole is given by a Gaussian law of error with a mean square deviation $\frac{d}{2}$, we get (cf. Mott and Massey 1933, p. 6), for not too large diffraction angles, an angle distribution which is again Gaussian with a standard deviation just represented by (1.3.1).

Since, for small values of the deflection angles, we have, according to (1.1.3),

$$\vartheta \approx \frac{b}{p}, \qquad (1.3.3)$$

we see from (1.3.1) that, provided $b \gg \lambda\!\!\!^{-}$, it is possible for any given value of the impact parameter p to choose d small compared with p and, at the same time, essentially to limit the diffraction to angles smaller than ϑ. Under such circumstances we may, at any rate approximately, visualize an orbit with a definite impact parameter. It is, of course, evident that the wave-length must be small compared with the collision diameter if classical mechanics shall be at all applicable to the calculation of the large deflection angles corresponding to small impact parameters, but the interest of the above considerations just lies in the proof that, for a

Coulomb field, this condition is sufficient for the approximate applicability of the classical calculation also for the smaller angles, corresponding to large impact parameters, which are statistically more frequent and, therefore, of decisive importance for many phenomena.

As regards the accuracy obtainable under given circumstances by an ordinary mechanical analysis, it must be taken into account that the finite size of the hole in the diaphragm implies an uncertainty in the fixation of the impact parameter, and the deflection angles to be expected according to (1.1.3) will, therefore, be distributed around a mean value with a standard deviation $\delta\vartheta$. By differentiating (1.3.3) and putting $\delta p = \frac{d}{2}$, we get thus

$$\delta\vartheta = \frac{b}{p^2}\frac{d}{2} = \frac{d}{2b}\vartheta^2. \qquad (1.3.4)$$

As a measure of the latitude $\Delta\vartheta$ in the deflection of the particles due to the combined effects of the diaphragm and the field, we may now take

$$\Delta\vartheta = \sqrt{(\delta\varphi)^2 + (\delta\vartheta)^2} \qquad (1.3.5)$$

and one thus finds, by means of (1.3.1) and (1.3.4),

$$\Delta\vartheta \geq \sqrt{\frac{\lambda}{b}}\,\vartheta \qquad (1.3.6)$$

for the degree of accuracy obtainable by a classical description of the phenomenon.

Introducing the notation

$$\varkappa = \frac{b}{\lambda} \qquad (1.3.7)$$

or, according to (1.1.4) and (1.3.2),

$$\varkappa = \frac{2\,|\,e_1 e_2\,|}{\hbar v}, \tag{1.3.8}$$

we thus have

$$\varkappa \gg 1 \tag{1.3.9}$$

as the necessary and sufficient condition for the justification of the classical considerations leading to the Rutherford formula (1.2.2). For decreasing values of $\varkappa$, orbital pictures gradually lose their applicability and, for values of $\varkappa$ of the same order of magnitude as unity or smaller, will have lost all physical significance.

An illustrative example of the inadequacy of orbital pictures is offered by the typical quantum-mechanical exchange effects in collisions between identical particles referred to in the former paragraph and where, according to formula (1.2.5), just for $\varkappa \gtrsim 1$ the scattering may for wide angular regions, according to the specific statistics, be essentially larger or smaller than would follow from simple mechanics. The apparent paradox that even for $\varkappa \gg 1$ the differential cross-section does not conform with the Rutherford law but, according to (1.2.5), oscillates rapidly with changing ϑ round the classical value, finds its solution in the fact that any attempt, by means of a suitable set of diaphragms, to separate the orbits of the colliding particles and thereby exclude exchange phenomena, will involve a diffraction which would prevent any observation of the quantum-mechanical anomalies in the scattering law (cf. Mott 1930). In fact, it will be seen that the uncertainty in deflection angle, given by (1.3.6) for $\varkappa \gg 1$, exceeds the angular intervals over which the cross-section (1.2.5) undergoes oscillations.

Similar considerations also cover the effects in collision problems of such specific quantum-mechanical particle properties as spin and magnetic moment which evade interpretation by classical pictures and which become of special significance when the particle velocities approach that of light. In this connection, it is of interest that the criterion (1.3.9) applies also in relativity theory. In fact, the only modification in the above simple analysis consists in the replacement of the rest mass by $m_0\gamma$, which has no influence on the result since, according to (1.3.8), the value of $\varkappa$ is independent of the mass. It may, however, be noted that the condition $\varkappa \gg 1$ cannot, for $v \sim c$, be realized, unless the charges e_1 and e_2 are considerably larger than the elementary unit of electricity.

For the following discussion of the penetration phenomena it is especially important that, unless condition (1.3.9) is fulfilled, it is impossible by means of classical pictures to draw any conclusion as regards the corrections to be expected in the Rutherford scattering law due to the deviation of the actual field from a pure Coulomb field as a consequence of screening effects.

§ 1.4. Modification of Rutherford Law in Screened Fields.

In order to examine the characteristic effects of a screening of the scattering field, we shall for the potential energy of the incident particle at distance r from the centre choose the simple expression

$$P_a(r) = \frac{e_1 e_2}{r} e^{-\frac{r}{a}}, \tag{1.4.1}$$

where a is a constant length which we shall refer to as the "screening parameter". A potential of this type covers, in fact, most of the screening problems of interest for our discussion and, in particular, (1.4.1) holds with high approximation for the electrostatic fields within atoms. The actual values of a in the various cases will be more closely discussed in the next chapter dealing with the applications of the present considerations to specific penetration phenomena.

As we shall see, the character of the problem will depend essentially on the ratio between a and the collision diameter

b for unscreened fields, defined by (1.1.4), and it will, therefore, be convenient to introduce the abbreviation

$$\zeta = \frac{b}{a}. \tag{1.4.2}$$

Now, in many of the most important penetration phenomena, $\zeta \ll 1$, and in such cases, which will be especially considered in this paragraph, the Rutherford law will in general be valid over a considerable angular region.

If condition (1.3.9) is fulfilled, and orbital pictures can be applied to collisions in a simple Coulomb field, the limit for the Rutherford scattering is easily estimated. In fact, in collisions with impact parameter p small compared with a, the deflection will occur practically only in the unscreened part of the field and will, therefore, with a high degree of accuracy, be given by (1.1.3). Consequently, we shall expect the scattering law (1.2.2) to hold approximately for angles larger than the value ϑ_a' obtained from (1.1.3) for $p = a$. Since, for $\zeta \ll 1$, this angle will be small, we get simply

$$\vartheta_a' \approx \zeta. \tag{1.4.3}$$

For greater values of p, the deflection angle ϑ will, due to the screening of the field, decrease far more rapidly than corresponding to (1.3.3).

As follows from considerations like those in § 1.3, this latter circumstance may prevent the unambiguous use of orbital pictures for $\vartheta \ll \vartheta_a'$, but, if (1.3.9) is fulfilled, this limit will evidently not be reached before the scattering distribution has become extremely rare in comparison with (1.2.2). For many purposes, we may, therefore, simply neglect the scattering for angles smaller than ϑ_a' and take this angle as the effective lower limit for the validity region of the Rutherford distribution.

For velocities close to that of light, the scattering formula will, as already mentioned, demand modifications especially as regards the close collisions, but it is of interest to note that, assuming ϑ'_a to be small, the relation (1.4.3) will hold if only, in (1.1.4), m_0 is replaced by $m_0\gamma$, implying, according to (1.4.2), a decrease in the value of ζ by a factor γ.

If (1.3.9) is not fulfilled, the effect of the screening presents us with a typical quantum-mechanical problem, the complete treatment of which depends on the solution of the appropriate wave equation. Still, to the purpose of the present discussion, it will be unnecessary to consider the exact solution but, as we shall see, we may confine ourselves to the first step of the Born approximation method. In fact, we shall be concerned only with a scattering effect so small that the plane wave representing the state of motion of the incident particle will pass practically unaltered through the field round the centre of force, and the diffraction can, therefore, in the usual simple way be described as a superposition of wavelets originating from all space elements round the centre. It is true that such a procedure does not converge for an unlimited Coulomb field, but this difficulty disappears just for the problem of screened fields in which we are interested.

For a field represented by (1.4.1), the simple method concerned leads, for an angle of deflection ϑ in relative coordinates, to the following expression for the amplitude A_s of the scattered wave at a large distance ϱ from the centre (cf. MOTT 1930 a, p. 25):

$$A_s(\vartheta) = \frac{A_i}{\varrho} \frac{e_1 e_2}{2 m_0 v^2} \cdot \frac{1}{\sin^2 \frac{\vartheta}{2} + \left(\frac{\lambda}{2a}\right)^2}, \qquad (1.4.4)$$

where A_i and $\lambda\!\!\!^-$ are the amplitude and the wave-length, divided by 2π, of the incident plane wave. As regards the statistical distribution of the deflection angles, we get, therefore,

$$d\sigma = R(\vartheta)\left[1+\left(\frac{\lambda\!\!\!^-}{2\,a\sin\frac{\vartheta}{2}}\right)^2\right]^{-2} d\omega, \qquad (1.4.5)$$

where $R(\vartheta)$ is defined by the Rutherford formula (1.2.2).

As already indicated, the approximation procedure leading to (1.4.5) is justified only if the scattering is so small that the incident wave passes without appreciable disturbance through the field of force. In order to examine how this condition depends on the charge and velocity of the incident particles, we may simply consider the total cross-section

$$\sigma = \pi a^2 \frac{\varkappa^2}{1+\left(\frac{\lambda\!\!\!^-}{2\,a}\right)^2} \qquad (1.4.6)$$

obtained by integrating (1.4.5) over all angles and introducing $\varkappa$ from (1.3.7). In fact, formula (1.4.6) shows that, if

$$\varkappa \ll 1, \qquad (1.4.7)$$

σ will, for all values of a, be small compared with πa^2, representing the number of particles which per unit time enter the unscreened part of the field. If (1.4.7) is fulfilled, only a small fraction of these particles is, therefore, deflected and the method leading to (1.4.5) will thus be valid.

As regards the relation between formula (1.4.5) and a simple Rutherford scattering, it is seen that only in the case where a is large compared with $\lambda\!\!\!^-$ does there exist an angular region for which $d\sigma$ conforms closely to the Rutherford law, while, for $a \ll \lambda\!\!\!^{-}$, we get a uniform scattering over all angles, as also follows directly from simple arguments. Further, we find that, for $a \gg \lambda\!\!\!^{-}$, the effective limit for the region of the Rutherford scattering may be taken as

$$\vartheta''_a \approx \frac{\lambda\!\!\!^{-}}{a} = \zeta \varkappa^{-1}. \qquad (1.4.8)$$

It is interesting to note that the estimates given by (1.4.3) and (1.4.8) give the same result for $\varkappa \sim 1$, in which case, strictly speaking, neither the application of classical orbital pictures nor the method of simplified wave diffraction is justifiable. Due to this remarkable fitting together, the two mutually exclusive procedures will, for $\zeta \ll 1$, practically cover all possibilities.

A comparison of the actual angular scattering distribution with the Rutherford law for a constant small value of ζ and for different values of $\varkappa$ is given in Fig. 3, in which the ratio ξ between the differential cross-sections for screened and unscreened fields is plotted against $\log \operatorname{cosec} \frac{\vartheta}{2}$. The curve R'_a represents the case $\varkappa \gg 1$, where ξ is practically equal to unity until, within a narrow region around $\vartheta = \vartheta'_a$, it falls to a vanishingly small value. The case $\varkappa \ll 1$ is represented by the curves R''_a, T, and S. In curve R''_a, we have again a region where ξ is practically unity, limited by a region around $\vartheta = \vartheta''_a$, where it rapidly vanishes. According to (1.4.3) and (1.4.8), the decline of R''_a is displaced relative to the decline of R'_a by the amount $\log \varkappa$. The steepness of the slope of the curves in the figure corresponds to $\zeta \sim 10^{-4}$ and the value of $\varkappa$ for curve R''_a is chosen to be about $\sqrt{\zeta}$. Curve T, which corresponds to $\varkappa \sim \zeta$ or

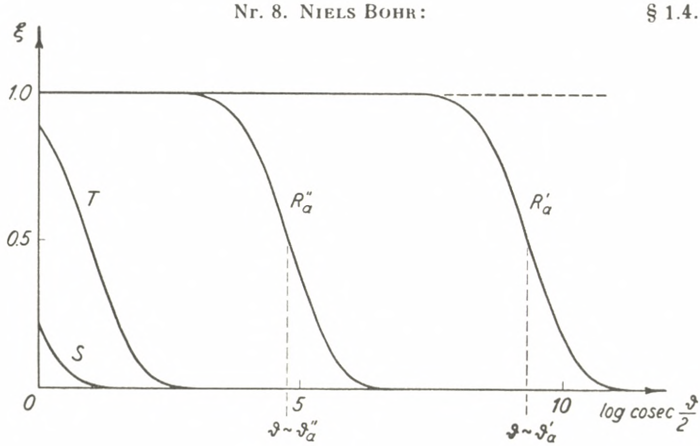

Fig. 3.

$a \sim \lambda$, represents the transition state where the scattering distribution for all angles begins to deviate essentially from the Rutherford law. Finally, curve S represents the spherically uniform angle distribution which occurs when $\varkappa$ is still smaller and, therefore, $a \ll \lambda$.

It need hardly be stressed that the results represented in Fig. 3 cannot be interpreted by even a restricted reference to orbital pictures. Thus, any attempt to attribute the difference between R'_a and R''_a to the obvious failure of such pictures in accounting for collisions with an impact parameter smaller than λ will be entirely irrelevant. In fact, this argument would imply a difference between the two distributions for the large angle scattering, while the actual differences occur only in the limits of small angles. For $\varkappa \ll 1$, the scattering law is, in fact, determined by all parts of the field in a manner completely foreign to any ordinary mechanical analysis. In particular, it is interesting to note that the central region of the field of dimensions comparable with b which, on classical mechanics, is responsible for all large angle scattering will, for $\varkappa \ll 1$, as seen from (1.4.6), by putting $a \sim b$, give rise to only a fraction of the order $\varkappa^4$ of the Rutherford scattering. Indeed, for a Coulomb potential, only regions of the field of dimensions comparable with or larger than $\lambda\!\!\!^-$ will contribute appreciably to the scattering effects.

Interesting illustrations of the characteristic features of the scattering problem for different values of $\varkappa$ are offered by the exchange phenomena appearing in collisions between identical particles and referred to in previous paragraphs. The problem is especially simple for particles with spin zero, where the initial stage of the collision in relative co-ordinates is represented by two scalar wave trains of equal amplitude and opposite direction, and the phases of which coincide at the centre of the field. Now, for $\varkappa \ll 1$, these waves will pass practically undisturbed through the field and, in any direction, the scattered waves from either of the wave trains will evidently have the same phase as the wavelets scattered from the centre. We shall, thus, expect the total scattering at an angle ϑ to be given by the square of the sum of the amplitudes corresponding to the deflections ϑ and $\pi - \vartheta$ of a single particle in a fixed field. This result is in accordance with formula (1.2.5) for vanishing values of $\varkappa$, which is just the quantity appearing in the last term of this formula. In case $\varkappa \gg 1$, the waves will not be able to penetrate the field, but will, on the contrary, be rather sharply deflected at a distance from the centre which, for angles of the order of $\frac{\pi}{2}$, is approximately equal to the classical collision diameter b. For deflections through exactly right angles, the two reflected waves will, of course, always be in phase but, for small deviations ε from this direction, a phase difference of the order $\varepsilon \frac{b}{\lambda\!\!\!^-} = \varepsilon\varkappa$ will appear, giving rise to the steep maxima and minima in the scattered intensity exhibited by formula (1.2.5).

Even in the problem of the scattering in a fixed field, anomalies related to the value of $\varkappa$ appear when the particle velocity approaches that of light. In fact, as referred to in § 1.1, a classical relativistic treatment gives peculiar effects for impact parameters of the order of magnitude of b while, for $\varkappa \ll 1$, no corresponding deviations from the Rutherford formula occur since, as mentioned above, the central part of the field at distances smaller than $\lambda\!\!\!^-$ has no significant influence on the scattering. In the latter case, the only correction, if we look apart from spin effects, consists in the replacement of m_0 by $m_0\gamma$ in the scattering law. According to (1.3.8), this correction does not affect the arguments leading to (1.4.7), but merely implies that in the estimate (1.4.8) the value of ζ is decreased by a factor γ.

§ 1.5. Problems of Excessive Screening.

In problems where $\zeta \ll 1$, representing the case of a Coulomb field with minor screening, we find, as shown in the preceding paragraph, in general a scattering which over a considerable angular interval conforms with the Rutherford law. If ζ is of the same order as or even larger than unity, however, we must for all values of $\varkappa$ expect an angular distribution which differs essentially from the scattering law in unscreened fields. In particular, the great frequency of small angle deflections will be far less pronounced and the scattering law may often approach a spherically symmetrical angular distribution given by

$$d\sigma = \frac{\sigma}{4\pi} d\omega, \tag{1.5.1}$$

corresponding to a total cross-section σ.

Also in such problems of excessive screening, the two approximation methods of classical mechanics and of simplified wave diffraction, respectively, apply with considerable accuracy to a large variety of problems, but these methods do no longer cover all possibilities to the same extent as for $\zeta \ll 1$. In fact, for $\zeta \gtrsim 1$, we must expect the regions where such approximations are applicable to be separated by an intermediate region where more general quantum-mechanical methods are required. The remarkable simplicity which the problems of minor screening exhibit originates in the circumstance that, only for a pure Coulomb field, the two in their physical interpretation incompatible, extreme approximation methods lead to the same statistical results for the collision effects.

To examine the conditions for the use of orbital pictures for a scattering field of the type (1.4.1), we may apply a procedure quite analogous to that used in § 1.3. Since, in the present case, the field intensity for all values of r varies essentially over a distance comparable with a, it is not sufficient that $\lambda\!\!\!^{-} \ll b$, but it must obviously be demanded that $\lambda\!\!\!^{-} \ll a$. Still, even if this condition is fulfilled, it is only over a limited angular region that the scattering can be accounted for by ordinary mechanics. In fact, in contrast to the case of Coulomb scattering where, for $\varkappa \gg 1$, orbital pictures are approximately applicable to all angles, it follows from (1.3.1) that any attempt by a suitable diaphragm to define the impact parameter with a latitude smaller than a will make it impossible to trace deflections smaller than

$$\vartheta^* \approx \frac{\lambda\!\!\!^{-}}{a} = \frac{\zeta}{\varkappa}, \tag{1.5.2}$$

according to (1.3.7) and (1.4.2). In many problems of excessive screening, however, the large angle deflections play a predominant part, as is the case when the simple scattering formula (1.5.1) applies. Under such circumstances, the estimate based on classical mechanics will be justified if $\vartheta^* \ll 1$ or, according to (1.5.2), if

$$\varkappa \gg \zeta, \tag{1.5.3}$$

a criterion which, just for $\zeta > 1$, is more restrictive than (1.3.9).

A well-known example, where classical pictures have a large region of applicability but where the breakdown of such methods for small deflection angles may still be significant for certain purposes, is offered by the scattering from an impenetrable sphere of radius ϱ large compared with the wave-length λ of the incident particles or radiation. While classical mechanics, like geometrical optics, in this case leads to a uniform scattering over all angles with an effective cross-section $\pi\varrho^2$, a proper wave-mechanical treatment discloses an additional scattering for angles comparable with λ/ϱ. In fact, the scattering cross-section for the small angles alone equals the scattering as calculated with neglect of the diffraction, and the total cross-section is, therefore, $2\pi\varrho^2$ (Massey and Mohr 1933; see also Wergeland 1945).

The contribution of this "shadow" effect may be very easily estimated from elementary considerations of ordinary optics (cf., e. g., Drude 1906, p. 207), according to which two supplementary systems of diaphragms give identical diffraction patterns. By substituting for the sphere a diaphragm with a circular hole of radius ϱ, it is, thus, immediately seen that the intensity of the diffraction pattern which is now formed by the waves penetrating the hole corresponds, for $\lambda \ll \varrho$, to a cross-section of just $\pi\varrho^2$. It may be added that this simple argumentation at the same time shows that the doubling of the cross-section is a phenomenon which is independent of the geometrical shape of the scattering body.

It is of interest that, just in the case $\lambda > a$, where orbital pictures fail completely, (1.4.7) is no longer a necessary condition for the use of the other extreme approximation method, that of simplified wave diffraction. In fact, it follows from (1.4.6) that the cross-section will be smaller than πa^2 if only $\varkappa < \lambda/2a$ or, according to (1.3.7) and (1.4.2), if

$$\varkappa \ll \sqrt{\zeta} \tag{1.5.4}$$

which, in the case of excessive screening where $\zeta \gg 1$, is essentially less restrictive than (1.4.7).

While, in cases where (1.5.4) is fulfilled, the differential cross-section is given by the general formula (1.4.5), it is, in the region (1.5.3) where classical mechanics applies, often convenient to compare the screened field with an n^{th} power potential

$$P_n(r) = \frac{k_n}{r^n}. \tag{1.5.5}$$

In fact, at distance r from the centre, (1.4.1) will, as seen from a logarithmic differentiation, vary in a way corresponding to (1.5.5) for

$$n = 1 + \frac{r}{a} \quad \text{and} \quad k_n = e_1 e_2 a^{n-1} \cdot \left(\frac{n-1}{e}\right)^{n-1}. \tag{1.5.6}$$

While, for $r \ll a$, we have, of course, with high approximation a Coulomb field, the influence of the screening will, thus, at larger distances imply a field intensity corresponding to (1.5.5) for ever increasing values of n.

In a number of applications, the part of the field around $r = a$ will be decisive for the deflections and, in this region, we have effectively $n = 2$. For such a field, the ordinary theory of central motion (cf., e.g., Thomson 1906, p. 371) gives, for the deflection corresponding to an impact parameter p, the expression

$$\vartheta = \pi \left| \left(1 + \frac{2\,k_2}{m_0 v^2 p^2}\right)^{-1/2} - 1 \right|. \tag{1.5.7}$$

In the case of attractive forces, $k_2 < 0$, the value of ϑ becomes infinite for $p = p_c$ given by

$$p_c^2 = \frac{2\,|k_2|}{m_0 v^2} \tag{1.5.8}$$

and, for $p < p_c$, the relative motion consists of a spiral orbit which, through an infinite number of revolutions, approaches the centre. Of course, there can be no question of applying such calculations rigorously to all values of p, since the field will be of the inverse cube type only in the region around $r = a$.

For small angles, we get from (1.5.7), by introducing k_2 from (1.5.6) and applying (1.1.4) and (1.4.2),

$$\vartheta \approx \frac{\pi}{2e} \zeta \frac{a^2}{p^2} \tag{1.5.9}$$

giving

$$d\sigma \approx \frac{\pi}{4e} \zeta a^2 \cdot \frac{d\omega}{\vartheta^3} \tag{1.5.10}$$

for the differential cross-section. It must, however, be remembered that such formulae do not apply to arbitrarily small angles. In fact, for $p \gg a$, the field will fall off more rapidly than corresponding to $n = 2$ and, moreover, classical mechanics can only, even in the case of $\varkappa \gg \zeta$, be applied to deflections larger than ϑ^* given by (1.5.2).

It is of importance that (1.5.10) varies less rapidly with ϑ than the Rutherford law and, in many applications, this circumstance implies that the influence of the small deflections will be negligible. In such cases, one may, according to (1.5.1 and 1.5.7), reckon with a total effective cross-section

$$\sigma \sim \pi p_c^2 = \frac{\pi}{e} \zeta a^2 \tag{1.5.11}$$

which, as seen from (1.5.8), is proportional to v^{-2}, while (1.2.2) varies like v^{-4}.

A survey of the scattering in n^{th} power potentials may be simply obtained from dimensional considerations. Thus, within the domain of validity of classical mechanics, the scattering cross-section must be proportional to b_n^2, where

$$b_n = \left(\frac{2\,|k_n|}{m_0 v^2} \right)^{\frac{1}{n}} \tag{1.5.12}$$

which, for $n = 1$, corresponds to (1.1.4) and which, in case of repulsive forces, gives the minimum distance of approach. We have, therefore,

$$d\sigma = b_n^2 \cdot f_n(\vartheta)\, d\omega \tag{1.5.13}$$

where, for increasing values of n, the angular distribution $f_n(\vartheta)$ will tend more and more towards a uniform scattering in all directions, corresponding to (1.5.1).

In quantum mechanics, on the other hand, the cross-section may besides the quantities appearing in (1.5.1) involve $\hbar$ and dimensional considerations are, in general, ambiguous. In cases, however, where the method of simple wave diffraction is applicable, the cross-section will, for a field given by (1.5.5), be proportional to k_n^2 and it thus follows directly from (1.5.12) that

$$d\sigma = b_n^{2n} \lambda\!\!\!^{-}{}^{2-2n} g_n(\vartheta)\, d\omega, \tag{1.5.14}$$

where $\hbar$ is involved through $\lambda\!\!\!^{-}$. Only in the case $n = 1$, therefore, can this scattering law conform with the classical expression.

As regards the comparison between the formulae (1.5.13) and (1.5.14), it may be noted that, while the approximation procedure leading to the latter formula, of course, gives the same result for attractive and repulsive fields, such similarity appears only for $n = 1$ in the classical calculations which, in case of attractive fields, even lead to singularities for $n \geqq 2$. On the other hand, the simple wave treatment gives, for a field of the type (1.5.5), convergent results only for $1 < n < 3$.

As regards the dependence of the scattering problem on ζ, we note that, according to classical mechanics, the minimum distance of approach in a field of the type (1.4.1) is given by

$$\frac{\varrho}{a} = \zeta \cdot e^{-\frac{\varrho}{a}}. \tag{1.5.15}$$

For $\zeta \sim 1$, this formula gives $\varrho \sim a$, and the main deflections therefore take place in the part of the field corresponding to $n = 2$. For larger ζ we find somewhat greater values of ϱ and, correspondingly, larger effective values of n. Still, the increase of ϱ with ζ will be only slow, and the effective cross-section will in such cases be practically independent of v, remaining comparable with πa^2.

In the case of excessive screening, there will, as already mentioned, exist an intermediary region in which neither orbital pictures nor simple wave diffraction will apply. In this region, corresponding, according to (1.5.3) and (1.5.4), to

$$\sqrt{\zeta} < \varkappa < \zeta, \qquad (1.5.16)$$

we must be prepared for quite new features of the scattering problem, which demand the application of more general quantum-mechanical methods.

The case of repulsive fields is relatively simple, because the wave representing the incident particles will only to a small degree be able to penetrate the potential barrier, rapidly rising at distance a. Since, in the region (1.5.16), we have $\lambda\!\!\!^{-} > a$, the scattering will be uniform over all angles, and the effective cross-section will over the whole region be of the order of magnitude πa^2, just corresponding, as we have seen, to the value at both limits $\varkappa = \zeta$ and $\varkappa = \sqrt{\zeta}$. An illustrative example of the inadequacy, in the intermediate region, of orbital pictures as well as of wave-mechanical perturbation methods, is afforded by the simple problem of the scattering from a quite impenetrable sphere of radius a of particles with wave-length large compared with a. As is well known, we get in this case a uniform scattering distribution corresponding to a total cross-section $4\pi a^2$, a result which follows directly from the boundary conditions in wave mechanics which, due to the vanishing of the wave function over the interior of the sphere, demands along the whole surface $r = a$ the same numerical values for the amplitudes of the incident plane wave train and the outgoing spherical wave representing the scattered particles.

For attractive forces, the situation is essentially different. Here, the wave function in the interior of the field will closely approximate that of an "atom" formed by two particles of charges e_1 and e_2 and reduced mass m_0. Now, the distance over which the wave function in the central part of the field reverts its phase is comparable with the "atomic radius" given by

$$r_0 = \frac{\hbar^2}{m_0 \,|\, e_1 e_2 \,|} \sim \frac{\lambda\!\!\!^{-2}}{b} = a \zeta \varkappa^{-2}. \qquad (1.5.17)$$

Although, in the region (1.5.16), we have $\lambda\!\!\!^{-} > a$, corresponding to $\varkappa < \zeta$, we see that nevertheless $r_0 < a$, since $\varkappa > \sqrt{\zeta}$, and the phase of the wave function will, therefore, undergo major variations between $r = 0$ and $r = a$. The possibility thus exists of the occurrence of peculiar quantum-mechanical resonance phenomena depending on the behaviour of the interior wave in the region around $r = a$, where it must be fitted together with the incident and outgoing waves. Under these circumstances, the scattering cross-section will vary strongly with the velocity of the incident particles and may in principle take on any value from zero to $4\pi\lambda\!\!\!^{-2}$.

Such effects are especially illustrated by the scattering of electrons by heavy atoms. Thus, the effective collision cross-section for slow electrons in inert gases was found by RAMSAUER (1923) to be vanishingly small compared with the geometrical cross-section of the atoms. The detailed treatment of the phenomena (FAXÉN and HOLTSMARK 1927) shows widegoing analogies to acoustical resonance problems. For larger electron velocities for which $\lambda\!\!\!^{-}$ becomes comparable with a, more complicated scattering effects showing characteristic maxima and minima in the angular scattering distribution have been observed (ARNOT 1931). These anomalies, which depend very sensitively on the electron velocity (WERNER 1931 and 1933), have their origin in departures from spherical symmetry of the wave function within the atoms, implying the appearance of zonal harmonics of higher order (Kalckar 1934). For still higher velocities where $\lambda \ll a$ and, thus, $\varkappa \gg \zeta$, we enter into the region where classical mechanics approximately applies.

§ 1.6. Survey of Collision Problems.

The situation as regards scattering in screened Coulomb fields is schematically represented in Fig. 4, where every point in the diagram represents a set of values of $\varkappa$ and ζ. For convenience, $\log \varkappa$ and $\log \zeta$ are chosen as coordinates. The oblique line T corresponds to $\varkappa = \zeta$ and the line U to $\varkappa = \sqrt{\zeta}$. Further, the horizontal and vertical hatchings indicate the regions for the applicability of classical mechanics and simplified wave scattering, respectively. The various angular regions denoted by R'_a, R''_a, S, P, and Q correspond to the different types of collisions discussed in the preceding paragraphs.

The vertical line L_1, for which ζ has a small constant value, is drawn as an illustration of the considerations in § 1.4, and the sections into which L_1 is divided by the abscissa axis and the lines T and U refer just to the cases indicated in Fig. 3, the curves T and R''_a corresponding to the points where L_1 cuts T and U, respectively.

The vertical line L_2 represents a large value of ζ corresponding to the problem of excessive screening discussed in § 1.5. While the section below the line U represents the region S of spherically uniform scattering treatable by the method of simple wave diffraction, and the section above the line T belongs to the region P in which, as indicated by the broken horizontal hatching, classical mechanics applies with certain restrictions, both simplified approxi-

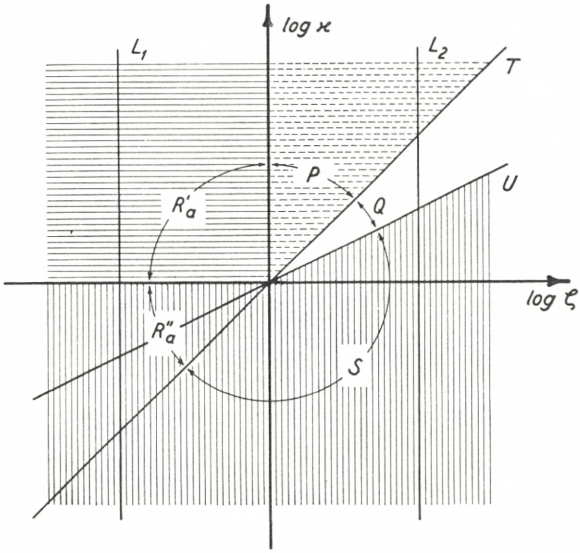

Fig. 4.

mation methods fail completely in the intermediate unhatched region Q, where quantum-mechanical resonance effects may occur.

As we shall see in the following, the regions R''_a and R'_a will especially apply to the problem of the penetration of fast particles like α- and β-rays and fission fragments, while the region P corresponds in particular to the situation met with for slower particles such as recoil atoms from radioactive disintegrations. For orientation, it may be remarked that in collision problems where all other parameters are fixed, a variation of the velocity will in the diagram correspond to a displacement along a line parallel to U, and it is seen that, while such lines for $\varkappa < \sqrt{\zeta}$ will run entirely in the region of vertical hatching, they will for $\varkappa > \sqrt{\zeta}$ pass through regions where simple wave diffraction and classical mechanics, respectively, apply as well as regions where both of these mutually exclusive methods fail.

CHAPTER 2

Penetration Phenomena Depending on Nuclear Collisions.

§ 2.1. Characteristics of Nuclear Collisions.

In the application to actual penetration phenomena of such simple calculations as those outlined in Chapter 1, it must, of course, be taken into consideration that, in general, collisions between atomic systems, built up of nuclei and electrons, present a complicated many-body problem. Due, however, to the large mass of the nucleus compared with that of the electrons we may, with a very high degree of approximation, distinguish between "nuclear collisions" in which momentum and kinetic energy are transferred to translatory motion of the stopping atom as a whole (elastic encounters), and "electronic collisions" in which energy is transferred to the individual electrons of the atoms, resulting in atomic excitation and ionization processes (inelastic encounters). The latter effects, which in many cases are primarily responsible for the stopping phenomena, will be examined in detail in Chapter 3, but here we shall first consider the part played in penetration problems by the comparatively simpler nuclear collisions in which the presence of the electrons in the atoms merely involves an electrostatic screening of the nuclear field of force.

In nuclear collisions, it will thus be justified to make use of a simplified atomic model in which a specification of the binding of the individual electrons is disregarded and an estimate of the charge distribution within the atom is obtained by an appropriate statistical procedure like that developed by THOMAS and FERMI. While in special problems it is often necessary to examine in detail this distribution and the resulting force fields, it will, for the general survey here attempted, suffice to recall that such methods lead, for a large part of the atomic region, to a potential just of the type (1.4.1) discussed in the former chapter. Expressing the screening distance a in terms of the "radius" of the hydrogen atom

$$a_0 = \frac{\hbar^2}{\mu \varepsilon^2}, \tag{2.1.1}$$

where μ and ε denote the electronic mass and charge, respectively, and writing

$$a = \frac{a_0}{s}, \tag{2.1.2}$$

one finds, as is well known, for an atom of charge number z,

$$s = z^{1/3} \tag{2.1.3}$$

as an approximate estimate of the screening. For heavy penetrating particles like fission fragments, which over the whole range carry a large number of bound electrons, the total screening effect will, of course, depend not only on the presence of the electrons in the stopping atoms, but also on the electronic screening of the incident particles themselves and, therefore, in general present a somewhat complicated problem. In many cases we may, however, still reckon with a potential (1.4.1) with a screening distance given by (2.1.2) with

$$s = \sqrt{z_1^{2/3} + z_2^{2/3}}, \tag{2.1.4}$$

where z_1 and z_2 denote the atomic numbers of the colliding atoms. In fact, this simple symmetrical expression may be shown to account roughly for the interaction between two charge distributions corresponding to potentials of the type (1.4.1) for s equal to $z_1^{1/3}$ and $z_2^{1/3}$, respectively.

In the treatment of nuclear collisions we may, thus, directly apply the considerations of Chapter 1 according to which, as illustrated in Fig. 4, the various types of collision problems are characterized by the values of the quantity $\varkappa$ and of the ratio ζ between the collision diameter b and the screening parameter a. Introducing the "velocity" of the electron in the hydrogen atom,

$$v_0 = \frac{\varepsilon^2}{\hbar}, \tag{2.1.5}$$

we get from (1.3.8)

$$\varkappa = 2\, z_1 z_2 \frac{v_0}{v} \tag{2.1.6}$$

and from (1.1.4), (1.4.2), (2.1.1), and (2.1.2)

$$\zeta = 2\, z_1 z_2\, s \frac{\mu}{m_0} \left(\frac{v_0}{v}\right)^2. \tag{2.1.7}$$

An elimination of v from these two expressions gives

$$\frac{\varkappa^2}{\zeta} = \frac{2\, z_1 z_2}{s} \frac{m_0}{\mu} \tag{2.1.8}$$

which, together with the above estimates for s, shows that $\varkappa$ is always larger than $\sqrt{\zeta}$ and, consequently, we meet in nuclear collisions only with problems corresponding to the region above the line U in Fig. 4. Looking, for the moment, apart from the case of electrons as incident particles, it is further seen from (2.1.4) and (2.1.7) that, unless $v \ll v_0$, we have $\zeta \ll 1$. In fact, even the greatest possible values of z_1 and z_2 will be more than compensated by the small value of μ compared with nuclear masses.

For fast particles over the major part of the range, the collisions are thus of the types denoted in Fig. 4 by R_a' and R_a'', according as $\varkappa > 1$ or $\varkappa < 1$, respectively. From (2.1.6) it is seen that, for $v < v_0$, we have always to do with the former case while, for larger velocities, it will depend on the values of z_1 and z_2 whether the condition $\varkappa > 1$ for the applicability of classical mechanics is fulfilled. In both cases, the scattering will conform with the Rutherford law for angles larger than a small value ϑ_a given by (1.4.3) and (1.4.8) for $\varkappa > 1$ and $\varkappa < 1$, respectively.

For particle velocities small compared with v_0 we may have $\zeta \gtrsim 1$, corresponding to a scattering which, for all angles, differs from the Rutherford law. Since, however, except for excessively low velocities, it follows from (2.1.6) and (2.1.7) that $\varkappa > \zeta$, the problem will belong to the region denoted in Fig. 4 by P and, as discussed in § 1.5, the collisions may thus be largely treated by means of classical mechanics. For increasing values of ζ, the scattering approaches a uniform distribution over all relative angles with a cross-section $\pi\varrho^2$, where ϱ is of the same order of magnitude as a. Still, it must be noted that for very slow particles such estimates as (2.1.3) or (2.1.4) lose their validity, since the colliding atoms will no longer be able to penetrate each other and, like in the kinetic theory of gases, we may for ϱ simply take the sum of the atomic radii, each being comparable with a_0.

If the incident particle is an electron, in which case $z_1 = 1$ and $m_0 = \mu$, we get from (2.1.7) and (2.1.3)

$$\zeta = 2\, z_2^{4/3} \left(\frac{v_0}{v}\right)^2. \tag{2.1.9}$$

For fast β-rays, for which $v \gg z_2^{2/3} v_0$, we have, therefore, $\zeta \ll 1$ and, like in the case of encounters between nuclear particles, the collisions will be of the types R_a' and R_a'' according as

$$\varkappa = 2\, z_2 \frac{v_0}{v} \tag{2.1.10}$$

is larger or smaller than unity. For smaller electron velocities, however, for which $\zeta \gtrsim 1$, the collisions will not, to the same extent as for heavy particles, be of the type P. In fact, already for $v < v_0 z_2^{1/3}$, in which case $\lambda\!\!\!^{-} > a_0 z_2^{-1/3} = a$, we have $\varkappa < \zeta$, corresponding to the region Q in Fig. 4 where, as described in § 1.5, characteristic quantum-mechanical resonance effects occur.

§ 2.2. Frequency of Individual Collisions. Branch Distribution.

The statistical laws governing individual collisions between atomic particles find direct application in the account of phenomena like the scattering of α-rays by thin foils of matter where the rays, except in those comparatively rare cases in which they suffer a considerable deflection in a single close nuclear collision, pass through the foils practically without change of direction or velocity. In general, however, it is necessary in penetration problems, besides individual effects of more violent collisions, to take into account the accumulative results of a very large number of collisions which, individually, produce only minor effects but which, together, are responsible for phenomena like the continual bending of the paths, termed compound scattering, and the gradual slowing down of the particles.

Such accumulative effects form the main topics of the following discussion but, in the present paragraph, we shall first consider a few problems depending on separate collisions. While, already in the former chapter, reference was made to the studies of large angle scattering, which have been so important as a means of exploring the structure of atoms, we shall here briefly consider the phenomenon of branch formation which occurs when an atom in the stopping material receives an energy sufficient to produce a visible trace in the cloud-chamber, and the frequency and distribution of which is a characteristic feature of the various types of penetrating particles.

From the considerations in § 2.1 it follows that, for fast particles for which $\zeta \ll 1$, the Rutherford law will hold over a considerable interval of the relative deflection angle ϑ and, specifying the branches by the angle ψ which they form with the stem and by the energy transfer T in the collision, we get within this region, from (1.1.7) and (1.2.1), for the differential cross-section expressed in terms of ψ

$$d\sigma = \frac{\pi}{2} b^2 \sin\psi \sec^3\psi \, d\psi \qquad (2.2.1)$$

and, from (1.1.8), (1.1.9), and (1.2.2), for the distribution of T,

$$d\sigma = B_\nu \frac{dT}{T^2}, \qquad (2.2.2)$$

where, the suffix ν referring to nuclear collisions,

$$B_\nu = 2\pi \frac{z_1^2 z_2^2 \varepsilon^4}{m_2 v^2}. \tag{2.2.3}$$

An integration of (2.2.2) gives

$$\sigma = B_\nu \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \tag{2.2.4}$$

for the cross-section for nuclear collisions with energy transfers between T_1 and T_2, provided that $T_2 < T_m$ and $T > T_a$, where T_m is the maximum value given by (1.1.9) and where T_a is the value of T corresponding to the limiting angle ϑ_a depending on the screening of the field. Since, under the circumstances assumed, ϑ_a is small, we have $T_a \ll T_m$ according to (1.1.8).

If we consider a particle penetrating a layer of matter of thickness ΔR, containing N atoms per cm^3, we can from (2.2.4) directly deduce the statistical distribution of the number of nuclear collisions suffered by the particle. In fact, assuming ΔR to be chosen so small that in almost every case the particle will penetrate the layer without appreciable change of its velocity, the average number of specified collisions is given by

$$\omega = N \Delta R \sigma = N \Delta R B_\nu \left(\frac{1}{T_1} - \frac{1}{T_2}\right). \tag{2.2.5}$$

Of course, the frequency of collisions will be subject to statistical fluctuations and the probability for the occurrence of just n collisions in an interval of T, where the mean number is ω, will be given by the well-known Poisson law

$$P(n) = \frac{\omega^n}{n!} e^{-\omega}, \tag{2.2.6}$$

which holds for any problem involving effects depending on a practically unlimited number of trials, for each of which the probability of occurrence is vanishingly small. In the present problem, the trials are represented by the fortuitous collisions with the nuclei of the atoms along a section of the path long compared with atomic dimensions.

The above formulae may be directly applied to fast particles like α-rays from radioactive disintegrations or nuclear fragments from fission processes. In fact, for $v \sim 5\, v_0$, formula (2.1.7) gives, for fission fragments ($z_1 \sim 50$), values of ζ of about 10^{-3} in all stopping materials and, for α-rays ($z_1 = 2$), values of the order of 10^{-4} or 10^{-3} depending on z_2. Since B_ν is proportional to z_1^2, formula (2.2.5) accounts, in particular, for the conspicuous difference in the frequency of branching observed for α-rays, where the phenomenon occurs for only a very small fraction of the particles, and fission fragments, where each track in general shows a large number of branches. By a statistical examination of the distribution of branches of specified length along the tracks of cloud-chamber pictures of fission fragments (Bøggild, Brostrøm and Lauritsen 1940) it has even been possible to draw conclusions from (2.2.5) regarding the velocity range relation. Moreover, the existence of two main groups of fragments with different masses and charges was borne out by the observation that the branch distribution of the tracks deviated markedly from (2.2.6), but could be well represented by a sum of two such expressions with different values for ω corresponding to the different charges and velocities of the particles of the two groups.

It is further of interest to note that the study of the branch distribution gives a very direct means of illustrating the difference between the Rutherford law and the more uniform scattering distribution to be expected for slower particles for which $\zeta \gtrsim 1$. In fact, from (1.1.7) and (1.5.1) we find

$$d\sigma = \sigma \sin 2\,\psi d\psi, \tag{2.2.7}$$

which differs essentially from (2.2.1), in particular for the less violent collisions for which $\psi \sim \frac{\pi}{2}$. While, thus, in fission fragment tracks, the great majority of the branches form right angles with the stem, a distribution of the type (2.2.7) was actually observed by Joliot (1934) in cloud-chamber studies of tracks of the much slower recoil atoms in α-ray disintegration. This last result is in accordance with the fact that, for such particles which have velocities of about $\frac{1}{10}\,v_0$, we get from (2.1.7) values of ζ of the order of unity.

In the region of uniform scattering, we further find from (1.1.8) and (1.5.1)

$$d\sigma = \sigma \frac{dT}{T_m} \tag{2.2.8}$$

and, for the number of collisions for which $T_1 < T < T_2$, we get, therefore,

$$\omega = N\Delta R\sigma \frac{T_2 - T_1}{T_m}. \tag{2.2.9}$$

In particular, this formula gives the total number of branches in a section ΔR of the range, if we put $T_2 = T_m$ and $T_1 = T_c$, where T_c is the smallest energy transfer to the gas nuclei giving rise to visible branching.

In problems where, as in collisions between hard elastic spheres, σ is independent of the relative velocity, we shall, according to (2.2.9), expect the branch frequency to decrease along the range as T_m diminishes and approaches T_c.

In his experiments, however, JOLIOT finds—apart from unexplained abnormal features of the pictures in the immediate neighbourhood of the point where the radioactive disintegration takes place—a steady increase in the number of branches along the track. This observation fits well with the expression (1.5.11) which, according to (1.5.8), gives σ proportional to v^{-2}, and which corresponds to the assumption that, in collisions between the recoil particles and the nuclei in the stopping material, the repulsive force varies approximately as the inverse cube of the distance. As mentioned in § 1.5, a field of force of this type is to be expected for $\zeta \sim 1$, a condition which has been seen to be roughly fulfilled for the α-recoil particles. In the following, it will be shown that the simple estimate of σ also accounts for the range velocity relation of these particles.

§ 2.3. Stopping Effects of Nuclear Collisions.

Apart from the large angle scattering and the track branching due to close nuclear collisions, a particle penetrating through matter will suffer a large number of less violent collisions the accumulative effect of which, as already mentioned, gives rise to the gradual stopping of the particle and to the compound scattering. Although, as we shall see, these stopping and scattering phenomena are intimately connected, it will be convenient, in order to bring out the statistical arguments as clearly as possible, first to consider the nuclear stopping problem in some detail.

To this purpose, we divide the collisions suffered by a particle along a given part of its range, over which its motion on the average is changed only little, into a large number of groups corresponding to small intervals of the energy transfer T. The total energy loss of the particle may, therefore, be written

$$\Delta E = \sum_i T_i n_i, \tag{2.3.1}$$

where n_i, the number of collisions in the i^{th} interval, will be distributed around its mean value ω_i according to formula (2.2.6).

The average value ΔE will, thus, be given by

$$\overline{\Delta E} = \sum_i T_i \omega_i \tag{2.3.2}$$

with a mean square deviation

$$\Omega^2 = \overline{(\Delta E - \overline{\Delta E})^2} = \sum_i T_i^2 \overline{(n_i - \omega_i)^2} = \sum_i T_i^2 \omega_i, \tag{2.3.3}$$

since from (2.2.6) it follows that the mean square deviation of n_i itself is just equal to ω_i. Passing to the limit of infinitesimal intervals for T, and introducing the differential cross-section $d\sigma$, we get, with the notation used in § 2.2,

$$\overline{\Delta E} = N \Delta R \int T d\sigma \tag{2.3.4}$$

and

$$\Omega^2 = N \Delta R \int T^2 d\sigma, \tag{2.3.5}$$

respectively, for the stopping power and its fluctuations within a small section ΔR of the range.

Considering first the case of fast particles for which $\zeta \ll 1$, the differential cross-section will be given by (2.2.2) for $T_a < T < T_m$. Since $T_a \ll T_m$ and since, moreover, the frequency of collisions with $T < T_a$ for decreasing T rapidly becomes negligible compared with (2.2.2), we may, in estimating the contribution of nuclear collisions to the atomic stopping power, to a first approximation, disregard the encounters for which $T < T_a$. Thus, we get from (2.3.4)

$$\overline{\Delta_\nu E} = N \Delta R B_\nu \log \frac{T_m}{T_a} \tag{2.3.6}$$

and from (2.3.5)

$$\Omega_\nu^2 = N \Delta R B_\nu T_m . \tag{2.3.7}$$

While the expression for $\overline{\Delta_\nu E}$ depends essentially on T_a, we have simply put $T_a = 0$ in the expression for Ω_ν, for which, in case $T_a \ll T_m$, the screening would only constitute a minor correction.

As regards the argument in the logarithmic term in (2.3.6), we have from (1.1.8) and by means of (1.4.3) and (1.4.8), respectively,

$$\frac{T_m}{T_a} = \left(\frac{2}{\vartheta_a}\right)^2 = \begin{cases} \left(\dfrac{2}{\zeta}\right)^2 & \text{for } \varkappa > 1 \\ \left(\dfrac{2}{\zeta}\varkappa\right)^2 & \text{for } \varkappa < 1 . \end{cases} \tag{2.3.8}$$

For highly charged particles like fission fragments, it is seen from (2.1.6) that $\varkappa > 1$ for all velocities in question, but for fast particles of smaller charge like protons, α-rays or electrons, we may, in light stopping materials, have $\varkappa < 1$. In case $\zeta \ll 1$, it follows, however, from (2.1.8) that the expression (2.3.8) will, even for $\varkappa < 1$, always be large compared with unity.

Just due to the large value of the logarithmic argument, formula (2.3.6) represents for $\zeta \ll 1$ a high degree of accuracy in spite of the cursory character of the approximations involved. In fact, any more detailed estimate of the distribution of T in the neighbourhood of T_a would only lead to a correction of the same order of magnitude as unity in the logarithmic term. Similarly, a closer investigation of the transition for $\varkappa \sim 1$ between the distribution functions R'_a and R''_a would, for $\zeta \ll 1$, give only a small correction to the estimate of the stopping effects. Moreover, we note that, as far as the logarithmic term is large, it will be very insensitive to velocity variations, although the argument depends on v. For high speed particles, the rate of energy loss in nuclear collisions will, therefore, according to (2.2.3), with high approximation vary like v^{-2}. As the velocity decreases and the argument of the logarithmic term gradually becomes smaller, $\overline{\Delta_\nu E}$ will, however, vary less rapidly with v. In such respects, the expression (2.3.7) for Ω_ν is particularly simple, being, according to (1.1.9), independent of the particle velocity.

In the case of slower particles for which $\zeta \gtrsim 1$, a rough estimate of the nuclear stopping effects may be obtained by assuming the scattering to be uniform over all solid angles, corresponding to the expression (2.2.8) for the differential cross-section. Since any deviation from this distribution for the weak collisions is of only minor importance for the estimate of $\overline{\Delta_\nu E}$ as well as of Ω_ν, we get

$$\overline{\Delta E_\nu} = \frac{1}{2} N \Delta R \sigma T_m \tag{2.3.9}$$

and

$$\Omega_\nu^2 = \frac{1}{3} N \Delta R \sigma T_m^2 \tag{2.3.10}$$

from (2.3.4) and (2.3.5), respectively.

In the particular case of $\zeta \sim 1$, where, according to (1.5.8) and (1.5.11), the cross-section σ is roughly proportional to v^{-2}, expression (2.3.9) gives a stopping power approximately independent of the velocity. In Chapter 5, we shall see that such a rate of energy loss actually accounts for the range velocity relation of α-recoil particles. In this connection, we may note that, just in the velocity region where $\zeta \gtrsim 1$, the nuclear collisions constitute the main source of energy loss, in contrast to the case of $\zeta \ll 1$, where electronic encounters often are predominant for the stopping effect. In the extreme case $\zeta \gg 1$, the value of σ will, according to the considerations in § 1.5, over a wide velocity interval remain of the same order of magnitude as the gas kinetic cross-sections, and (2.3.9) therefore gives a stopping power proportional to v^2. Such conditions are to be expected for β-recoil particles where, for medium atomic numbers of the radioactive substance, we have $v \sim 10^{-2} v_0$ and $\zeta \sim 10^3$.

§ 2.4. Statistics of Nuclear Stopping Effects.

The elementary penetration theory for nuclear stopping effects, as outlined in § 2.3, allows directly an estimate of the mean value $\overline{\varDelta E}$ and the mean square deviation Ω^2 of the energy loss suffered by a particle traversing a certain thickness of matter. If the values of $\varDelta E$ are distributed according to a normal law of error

$$W_0(\varDelta E) = \frac{1}{\sqrt{2\pi}\,\Omega_0} e^{-\frac{(\varDelta E - \varDelta E_0)^2}{2\Omega_0^2}} \tag{2.4.1}$$

with maximum $\varDelta E_0$ and half width Ω_0, the theory thus gives a comprehensive account of the phenomenon since, of course, we have simply $\varDelta E_0 = \overline{\varDelta E}$ and $\Omega_0 = \Omega$.

To what extent the distribution of $\varDelta E$ will actually be given by a formula of the type (2.4.1) will, in the first instance, depend on the thickness of the matter penetrated. Thus, for very thin layers, the stopping effect will be quite irregular, depending on only a few collisions. However, even if ΔE is the result of a very large number of individual contributions, it is well known from probability theory that a normal law of error can be expected only if all the individual contributions are small compared with Ω, or if

$$\Omega > T_m, \tag{2.4.2}$$

where T_m is the maximum value of the energy transfer in an individual collision. If this condition is not fulfilled, a single encounter may, in fact, have an appreciable influence on the phenomenon, and the distribution may then deviate essentially from a Gaussian law. Under these circumstances, the most probable value of ΔE, which in general is the quantity directly obtainable from measurements, need no longer coincide with the mean value $\overline{\Delta E}$ and, similarly, the range straggling will not be essentially determined by the value of Ω. In such cases, therefore, an interpretation of experiments requires a more detailed analysis of the statistical distribution of ΔE.

A problem of this kind was first met with in the stopping of β-rays where, in electronic collisions, $m_1 = m_2 = \mu$ and where, therefore, values of T may occur which equal the total kinetic energy E of the incident particle. Under such circumstances, the condition (2.4.2) is evidently not fulfilled even for thicknesses of matter comparable with the whole range. In the discussion of the distribution of ΔE for a beam of β-rays a distinction was accordingly made (Bohr 1915) between the great majority of the particles, which suffer only smaller collisions and the stopping of which is a typical accumulative effect, and the few particles for which the value of ΔE is mainly determined by single violent collisions.

An approximate treatment was obtained by distinguishing between collisions for which T is smaller or greater than a certain value T^* defined in such a way that the particles on the average suffer about one collision for which $T > T^*$. As regards the collisions for which $T < T^*$, it was shown that the resultant energy loss $\Delta^* E$ is distributed roughly according to a Gaussian law, while the sporadic occurrence of the more violent collisions primarily gives rise to a "tail" in the distribution of ΔE extending far beyond the width of the Gaussian peak. A detailed analysis of the problem (Williams 1929) has given results which conform remarkably well with those obtained by such approximative considerations and we may, therefore, here confine ourselves to the more cursory procedure[1].

In order to examine to what extent condition (2.4.2) for a Gaussian distribution of the energy losses is fulfilled for nuclear stopping effects, we shall first consider the most important problems, for which $\zeta << 1$. In this case, it follows from (2.3.7) that (2.4.2) is equivalent to

$$N \Delta R B_\nu > T_m \tag{2.4.3}$$

or, according to (2.3.6), to

$$\overline{\Delta_\nu E} > T_m \log \frac{T_m}{T_a}. \tag{2.4.4}$$

[1] Added in proof: The writer's attention has been called to an article by L. Landau (Journ. of Phys. U. S. S. R. 8, 201 (1944)), who has essentially refined the theory by a more rigorous mathematical treatment of the problem, by which he has succeeded in obtaining an analytical expression for $W(\Delta E)$.

The contribution of nuclear collisions to $\overline{\Delta E}$ can, of course, even for values of ΔR comparable with the whole range, never be greater than $E = \frac{1}{2} m_1 v^2$ and, due to the energy transfer to the electrons, it will in fact often remain essentially smaller. Since, further, in the case considered, the logarithm is great compared with unity, we see, therefore, that it is a necessary condition for the fulfilment of (2.4.4) that $T_m \ll E$ or, according to (1.1.9), that m_1 is either very large or very small compared with m_2. Only in the case of $m_1 \gg m_2$, however, we have to do with a simple stopping phenomenon since, for $m_1 \ll m_2$, as we shall see in the next paragraph, the scattering will, just under the circumstances where (2.4.3) is fulfilled, be so large that we meet with typical diffusion effects. If $m_1 \sim m_2$, the situation is quite analogous to the problems of the stopping of high speed electrons in electronic collisions where, as mentioned above, we meet with a more composite statistical distribution of the resultant energy losses.

In the analysis of the distribution of $\Delta_\nu E$ in cases where (2.4.2) is not fulfilled, we shall distinguish between collisions for which T is smaller or larger than the value T^* defined by

$$T^* = N \Delta R \, B_\nu \tag{2.4.5}$$

which is smaller than T_m just in those cases where (2.4.3) does not hold. Since we shall assume that the total average number of collisions, given by (2.2.5) for $T_1 = T_a$ and $T_2 = T_m$, is large, it further follows that $T_a \ll T^*$.

Considering first the effect of the collisions for which $T < T^*$, we have for the mean value of the energy loss $\Delta_\nu^* E$ due to these encounters, in analogy to (2.3.6),

$$\overline{\Delta_\nu^* E} = N \Delta R \, B_\nu \log \frac{T^*}{T_a} \tag{2.4.6}$$

and, corresponding to (2.3.7), by means of (2.4.5),

$$\Omega_\nu^* = N\Delta R\, B_\nu = T^*. \tag{2.4.7}$$

Since $T^* \gg T_a$ we see from (2.4.6) and (2.4.7) that $\overline{\Delta_\nu^* E} \gg \Omega_\nu^*$ and since, moreover, as follows from (2.4.7), none of the individual contributions to $\overline{\Delta_\nu^* E}$ is larger than Ω_ν^*, we may conclude that, to a first approximation, the distribution of $\Delta_\nu^* E$ will be given by a normal law of error corresponding to (2.4.1) for a value of ΔE_0 closely equal to $\overline{\Delta_\nu^* E}$ and with a width Ω_0 of the same order of magnitude as Ω_ν^*.

As regards the collisions for which $T > T^*$, their average number will, according to (2.2.5) and (2.4.5), be smaller than unity, although approaching this value in the case $T^* \ll T_m$ where the condition (2.4.3) is far from being fulfilled and where the distribution of $\Delta_\nu T$ thus deviates essentially from a Gaussian law. Under these circumstances, some of the particles will suffer an energy loss several times greater than T^* and, although the probability of such violent collisions is very small, it will be far greater than the probability of the energy loss exceeding $\overline{\Delta_\nu^* T}$ by a corresponding amount, as a result of accumulative effects with $T < T^*$. The great majority of the particles will, however, either suffer no collisions with $T > T^*$ or will suffer one or a few collisions, each with a value of T only little larger than T^*. It follows, therefore, that the distribution of $\Delta_\nu E$ consists of a narrow peak and a flat tail extending far beyond the width of the peak. Furthermore, it is seen that the peak, although slightly unsymmetrical, resembles the Gaussian distribution of $\Delta_\nu^* E$. In particular will the half width of the peak be comparable with Ω_ν^*, and the position of the peak maximum will differ from $\overline{\Delta_\nu^* E}$ only by an

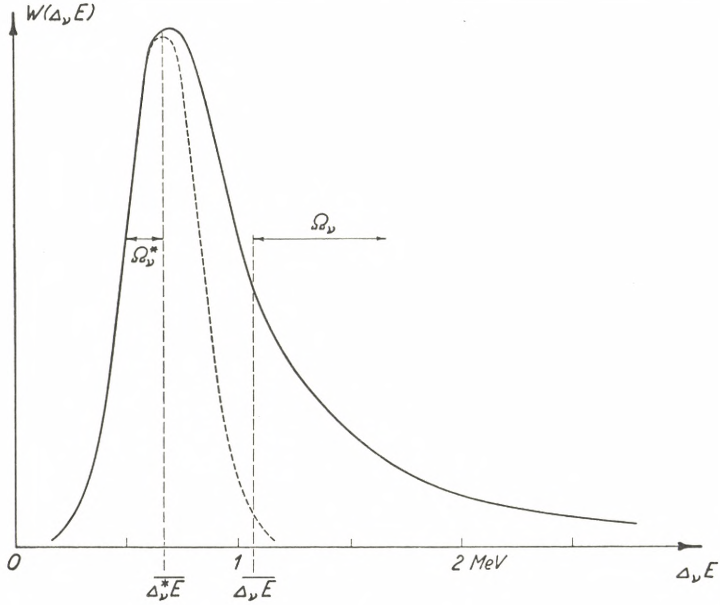

Fig. 5.

amount of the same order of magnitude as Ω_ν^*. As regards the shape of the tail, it follows from the distribution law (2.2.2) for individual energy losses that, for values of $\Delta_\nu E - \overline{\Delta_\nu^* E}$ large compared with Ω_ν^*, the probability function $W(\Delta_\nu E)$ will decrease as the inverse square of $\Delta_\nu E - \overline{\Delta_\nu^* E}$.

A typical straggling curve is illustrated in Fig. 5, which shows the distribution of $\Delta_\nu E$ to be expected for fission fragments of the light group ($Z \sim 38$, $A \sim 96$) with $v \sim v_0$ passing through 2 mm argon at N.T.P., roughly corresponding to half the residual range. In this case, we have $\zeta \sim 0.09$, as seen from (2.1.7), and, consequently, from (2.3.8), since $\varkappa \gg 1$, a value of $\frac{T_m}{T_a}$ of about 500. From (1.1.9) and (2.2.3) it further follows that $N \Delta R\, B_\nu \sim 0.08\ T_m$ so that, from (2.4.5), we have $40 T_a \sim T^* \sim 0.08\ T_m$. The distribution of $\Delta_\nu E$ is represented by the full-drawn curve, while $W(\Delta_\nu^* E)$ is indicated by the broken curve. As regards details of these curves, use has been made of the more refined analysis by WILLIAMS (1929). The most probable value of $\overline{\Delta_\nu E}$, represented by the position of the maximum of $W(\Delta_\nu E)$, coincides approximately with $\overline{\Delta_\nu^* E}$ while, as shown in the figure, $\overline{\Delta_\nu E}$ is considerably larger, due to contributions from the tail of the distribution curve. In the case illustrated, $\overline{\Delta_\nu^* E}$ is less than $\overline{\Delta_\nu E}$ by about 40 %. Further, the half width of the peak is seen to be of the same order of magnitude as Ω_ν^*, whereas Ω_ν, the value of which is also indicated, bears no such simple relation to the curves.

In contrast to the case $\zeta \ll 1$, where the great majority of the collisions lead to energy losses very small compared with T_m, the various values of T are, according to (2.2.8), for $\zeta \gtrsim 1$ equally probable and the distribution of the resultant energy losses, therefore, has an essentially different character. In this connection, we shall consider only the case $m_1 \gg m_2$ since, for $m_1 \lesssim m_2$, the incident particle in almost every collision will suffer a large deflection and the stopping and straggling phenomenon thus be overshadowed by diffusion effects. If, however, $m_1 \gg m_2$, in which case, as seen from (1.1.9), the particles can lose only a very small fraction of their energy in a single collision, the values of $\Delta_\nu E$, corresponding to a section of the range for which the average number of collisions is large, will be distributed according to a normal law of error. In fact, if $N \Delta R \sigma$, which represents the number of collisions, is large compared with unity, it follows from (2.3.10) that $\Omega \gg T_m$, corresponding to condition (2.4.2).

§ 2.5. Compound Scattering.

It has already been mentioned that the phenomenon of compound scattering, which is due to the accumulative effect of a large number of small individual deflections, is closely related to the nuclear stopping effects. As we shall see, the compound scattering offers, in fact, often a direct means of estimating the contribution of nuclear collisions to the total stopping power of the substance.

In cases where a separation between the large angle scattering and the compound scattering is possible, the majority of the deflections φ in individual collisions will be very small and, provided the mean square of the resultant angle Φ through which a particle is scattered by passing through a sheet of matter is also small, we, therefore, have

$$\Psi^2 = \overline{\Phi^2} = \sum_i \varphi_i^2 \omega_i, \tag{2.5.1}$$

where the collisions have been divided into groups corresponding to small intervals of φ and where ω_i, as in (2.3.2), denotes the average number of collisions in the i^{th} interval. Introducing the differential cross-section $d\sigma$, we may also write

$$\Psi^2 = N \Delta R \int \varphi^2 \, d\sigma \tag{2.5.2}$$

in analogy to (2.3.4).

In order to stress the intimate relationship to the stopping phenomena, it will be convenient to express the deflection φ suffered in a collision in terms of the energy loss T and, in this connection, it will be simplest first to consider the case $m_1 \gg m_2$, where only small values of φ can occur. Thus, from (1.1.6) we have approximately

$$\varphi = \frac{m_2}{m_1} \sin \vartheta \tag{2.5.3}$$

or, by introducing T from (1.1.8),

$$\varphi^2 = \frac{4\, m_2^2}{m_1^2} \frac{T}{T_m} \left(1 - \frac{T}{T_m}\right). \tag{2.5.4}$$

From (2.5.2) we find, therefore, by means of (2.3.4) and (2.3.5),

$$\Psi_\nu^2 = \frac{4\, m_2^2}{m_1^2} \left(\frac{\overline{\Delta_\nu E}}{T_m} - \frac{\Omega_\nu^2}{T_m^2} \right), \tag{2.5.5}$$

where, as in the preceding paragraphs, the suffixes ν indicate that we are concerned with the effects of nuclear collisions.

In the case of minor screening ($\zeta \ll 1$), the ratio between the first term and the second term within the brackets of (2.5.5) is equal to $\log \frac{T_m}{T_a}$, as seen from (2.3.6) and (2.3.7). Since $T_m \gg T_a$, we may, to a first approximation, neglect the second term and get from (1.1.9), neglecting m_2 compared with m_1,

$$\Psi_\nu^2 = \frac{m_2}{m_1} \frac{\overline{\Delta_\nu E}}{E}. \tag{2.5.6}$$

For excessive screening ($\zeta \gtrsim 1$), where the formulae (2.3.9) and (2.3.10) for $\overline{\Delta_\nu E}$ and Ω_ν^2 are to be applied, we get, instead,

$$\Psi_\nu^2 = \frac{1}{3} \frac{m_2}{m_1} \frac{\overline{\Delta_\nu E}}{E}, \tag{2.5.7}$$

since the second term within the brackets in (2.5.5) is equal to two thirds of the first term. In particular, it follows from these expressions for Ψ_ν that, even if $\overline{\Delta_\nu E}$ is comparable with E, the resultant deflections will, for $m_1 \gg m_2$, actually be small, as is presupposed in (2.5.1).

Like for the energy losses considered in § 2.4, it will in general be necessary to investigate in some detail the statistical distribution of the scattering angles and, to this purpose, we may use a criterion similar to (2.4.2). Since the largest individual deflection angle φ_m, according to (2.5.3), is equal to $\frac{m_2}{m_1}$, it follows from (2.5.6) or (2.5.7), respectively, that the compound scattering will be of the Gaussian type if $\frac{\overline{\Delta_\nu E}}{E}$ is large compared with $\frac{m_2}{m_1}$. While, of course, for very thin layers of matter corresponding to only few collisions we can expect no such simple statistical regularities, we see that, in case $\overline{\Delta_\nu E} \sim E$, the distribution of Φ will, for $m_1 \gg m_2$, actually correspond to a normal law of error with a mean square deflection angle equal to Ψ_ν^2.

If m_1 is comparable with or smaller than m_2, large individual deflections may occur, and the phenomenon will primarily depend on the relative frequency of large and small deflection angles. As already indicated in the former paragraph, we meet in this respect with an essential difference between problems for which $\zeta \ll 1$ and those where $\zeta \gtrsim 1$. In fact, in the latter case, the distribution of ϑ will be nearly uniform over all angles, and a considerable fraction of the collisions will result in large deflections φ. If the average number of collisions within the section of the range considered is large, we therefore have to do with a complete diffusion of the incident beam of particles. In the case of $\zeta \ll 1$, it follows from (2.2.5) that large deflections will be frequent or rare, according as (2.4.3) is fulfilled or not. While, in the former case, we have again to do with a typical diffusion problem, it will in the latter case in general be possible to separate between "single" and "compound" scattering.

This last problem, which has been especially examined by Bothe (1921) and by Williams (1939 and 1940), presents a close analogy to the problem of the energy losses discussed in the preceding paragraph. In conformity with the procedure used in distinguishing between the peak and the tail distribution of $\Delta_\nu E$, we may thus, in the scattering problem, separate between the effects of deflections smaller and larger than the value φ^* which corresponds to a collision with energy transfer T^*.

An estimate of the compound scattering in nuclear collisions for which $m_1 \gtrsim m_2$ may under such circumstances be obtained in a similar way as that used in calculating Ψ_ν for $m_1 \gg m_2$. Since the deflection angles corresponding to collisions in which $T < T^*$ are small, we have from (1.1.6) with a high approximation

$$\varphi = \frac{m_2}{m_1 + m_2}\vartheta \qquad (2.5.8)$$

and, thus, from (1.1.8) and (1.1.9),

$$\varphi^2 = \frac{m_2}{m_1}\frac{T}{E}. \qquad (2.5.9)$$

Provided the resultant spatial deflections due to individual collisions for which $\varphi < \varphi^*$ also remain small, we get from (2.3.4) and (2.5.2) for the mean square of these deflections

$$(\Psi^*_\nu)^2 = \frac{m_2}{m_1}\frac{\overline{\Delta^*_\nu E}}{E}, \qquad (2.5.10)$$

where $\overline{\Delta^*_\nu E}$ is the total average energy loss due to the collisions in question. Since, from the considerations in § 2.4, it follows that $\overline{\underset{\nu}{\Delta^*}E} \gg T^*$ if the number of collisions is large, we see from (2.5.9) and (2.5.10) that, in this case, $\Psi^*_\nu \gg \varphi^*$. Consequently, it is well justified to distinguish between a Gaussian distributed compound scattering and a single scattering given by the Rutherford law.

Introducing formula (2.4.6) for $\overline{\underset{\nu}{\Delta^*}E}$, we get from (2.5.10) by means of (2.4.5) an expression for Ψ^*_ν which closely coincides with that deduced by Bothe (1921) and applied by him especially in the study of the compound scattering of α-rays in different materials. The problem of the scattering of β-rays has been considered by Bothe (1923), and in particular by Williams (1939 and 1940), who has investigated in great detail also the contribution of individual scattering angles larger than φ^* to the average resultant deflection. In this connection, it is of particular interest to note that just the phenomenon of compound scattering offers a direct test of the different effects of the screening of the nuclear field to be expected for $\varkappa < 1$ and $\varkappa > 1$, respectively. In fact, as pointed out by Williams (1945), the measurements of the scattering of α-particles in heavy materials where, according to (2.1.6), the value of $\varkappa$ is large, are found to conform with the calculations based on classical mechanics while, for fast β-rays, where $\varkappa < 1$, the simple wave-mechanical treatment is consistent with experimental results.

While, in problems of nuclear stopping, we need not enter on relativity effects since, for very large velocities, electronic stopping completely predominates, it is of importance, as regards the compound scattering, to consider the corrections to be introduced for $v \sim c$. This problem, which has been treated in detail by Williams (1939), is rendered comparatively simple by the circumstance that the very violent collisions in which specific relativity effects occur have no great influence on the compound scattering. The primary modification to be introduced in the above formulae arises, therefore, from the increased inertia of the particle and is accounted for by simply replacing m_1 by $m_1\gamma$ in (2.5.6) and (2.5.10). Still, for very fast particles where the wave-length becomes smaller than nuclear dimensions, it is further necessary to take into account the deviations from the Coulomb scattering due to the finite size of the nucleus. As shown by WILLIAMS (1939), the latter effect, which may be treated on the same lines as the screening problems considered in § 1.4, implies a cut-off of the Rutherford distribution for angles larger than a certain limiting value.

The close relationship between compound scattering and stopping effects in nuclear collisions gives, as already indicated, a means of estimating the part played by such collisions in the penetration phenomena. For this purpose it is essential that electronic collisions, although often of determining influence on the stopping, are usually of only secondary importance for the scattering. In fact, for very fast particles, the contribution of each atomic electron to Ψ^2 will, apart from minor differences in the logarithmic term, be given by an expression corresponding to (2.5.6) or (2.5.10) for a nucleus of unit charge, and the resultant value of Ψ^2 may, therefore, approximately be obtained from these expressions by simply replacing z^2 by $z^2 + z$. For smaller particle velocities where, as follows from the considerations in Chapter 3, the state of binding of the atomic electrons is to a lesser degree influenced by the collisions, the effect of these electrons is essentially confined to the screening of the nuclear field, which is already included in the estimates of Ψ_ν.

From measurements of the compound scattering and of the energy decrease in a given section of the range, it is thus possible by means of the equations (2.5.6), (2.5.7) or (2.5.10) directly to determine the contribution of nuclear collisions to the stopping effect. In particular, the well-known fact that the tracks of fast α-particles, except at the very end of the range, form almost straight lines corresponds to the circumstance that nearly the whole energy loss is due to electronic collisions. In the case of fission fragments, however, the study of the conspicuous bending of the tracks in cloud-chamber pictures (Bøggild, Brostrøm and Lauritsen 1940) has given evidence that the nuclear stopping effect is in no way insignificant and even becomes predominant in the last part of the range where the velocity is comparable with v_0. To such problems we shall return in Chapter 5 in connection with a closer discussion of the bearing of the considerations in this chapter on the relative contribution of nuclear and electronic collisions to the stopping and straggling phenomena.

CHAPTER 3

Atomic Excitation and Ionization in Penetration Phenomena.

§ 3.1. Treatment on Classical Mechanics of Ionization and Electronic Stopping Effects.

In collisions between atoms and swiftly moving particles, there will, as already mentioned, besides the transfer of energy and momentum from the incident particle to the atom as a whole, also be an interaction between the particle and the individual atomic electrons to take into account. This interaction, which often is the main source of the stopping effect, may, in fact, result in an ionization or excitation of the atoms along the path. As already indicated in the Introduction, the treatment of these phenomena has been an important test of the methods of atomic mechanics and has not least offered instructive lessons as regards the extent to which the application of classical mechanical concepts is adequate and at what points proper quantum-mechanical analysis is required. In order to bring out the principal arguments as clearly as possible, it will be convenient to recall the outlines of the development by which the situation has been gradually clarified and, to this purpose, we shall first consider the problem from the standpoint of ordinary mechanics.

The subject was originally taken up by J. J. Thomson (1906) who, several years before the discovery of the atomic nucleus, discussed the stopping of high speed particles due to collisions with the electrons in the atoms of the matter penetrated. Assuming that the velocities of these electrons are small compared with the particle velocity, and neglecting in the first place the effect of the interatomic forces during the encounter, Thomson deduced, by simple mechanical considerations like those referred to in § 1.1, an expression for the statistical distribution of the individual energy losses equivalent to (2.2.2). With our notation, the differential cross-section for collisions between an atomic electron of charge ε and mass μ initially considered at rest, and a particle of charge number z_1 and velocity v was, thus, found to be

$$d\sigma = B_\varepsilon \frac{dT}{T^2}, \qquad (3.1.1)$$

where

$$B_\varepsilon = 2\pi \frac{z_1^2 \varepsilon^4}{\mu v^2}, \qquad (3.1.2)$$

the suffix ε standing for electronic collisions as distinct from nuclear collisions considered in Chapter 2.

In particular, formula (3.1.1) was used by Thomson (1912) to estimate the ionization effect of fast particles on the assumption that ionization occurs if the energy transfer to an electron exceeds the so-called ionization potential. If the various electrons in each atom are specified by an index s, and the energy which is necessary to remove the s^{th} electron from the atom is denoted by I_s, the average number of ions produced should, therefore, corresponding to (2.2.5), be given by

$$\omega_I = N \Delta R\, B_\varepsilon \sum_s \left(\frac{1}{I_s} - \frac{1}{T_m}\right), \qquad (3.1.3)$$

where the summation is to be taken over all atomic electrons for which the ionization energy I_s is smaller than T_m, the largest possible energy transfer in free collisions. From (1.1.9), we have

$$T_m = \left(\frac{1}{4}\right) 2\,\mu\, v^2, \qquad (3.1.4)$$

where the italic factor in brackets is to be omitted for heavy penetrating particles ($m_1 \gg m_2 = \mu$) and to be included only if the incident particle is an electron ($m_1 = m_2 \doteq \mu$). This way of combining the formulae for the various types of particles we shall apply throughout in the following.

The average energy $(\overline{\Delta_\varepsilon E})_I$ spent in ionization processes is, on the above assumptions, given by

$$(\overline{\Delta_\varepsilon E})_I = N \Delta R B_\varepsilon \sum_s{}' \log \frac{T_m}{I_s}, \qquad (3.1.5)$$

obtained from (2.3.4) and (3.1.1) by summing, as in (3.1.3), over the atomic electrons for which $I_s < T_m$. According to classical mechanics, we must, however, expect that energy may be transferred also in collisions which do not result in the removal of an electron from the atom. Still, it is evident that, in estimating this energy transfer, it is not permissible entirely to disregard the interatomic field of force, since the integral (2.3.4) diverges for vanishingly lower limit of T. In THOMSON's original treatment, it was assumed that formula (3.1.1) would hold only for values of T corresponding to impact parameters smaller than the average distance between the electrons in the atom while, due to neutralization effects, the average energy transfer was considered to be negligibly small in the more distant collisions. In a subsequent attempt by DARWIN (1912) to adapt the theory to the model of the nuclear atom, it was simply assumed that only collisions in which the particle penetrates the interior of the atom will contribute to the energy loss.

While the exchange of momentum between the particle and the atom as a whole is limited by the screening of the nuclear fields by the surrounding electrons, the limit of free energy transfer between the particle and the atomic electrons, however, will not depend primarily on the spatial charge distribution within the undisturbed atom. In fact, the stopping effect of the individual atomic electrons depends, in the first instance, on their displacement within the atom during the encounter. Notwithstanding the determining influence of the atomic forces in case $T < I_s$ on the state of binding of the electron after the collision, we must on ordinary mechanics expect that the binding forces cannot appreciably affect the energy transfer if the interaction between the particle and the electron is practically confined to a time interval short compared with the atomic oscillation period. Encounters of duration long compared with the atomic period, however, will practically have the character of an adiabatic process in which, at any moment, the atom may be regarded as exposed to a static field and, if the displacement of the electron during the encounter does not lead to its removal from the atom, its state of binding will after the collision be the same as before.

The atomic binding forces, thus, introduce, as regards the energy transfer, a kind of screening not of static, but of dynamic origin. In estimating the effect of this screening, we may compare the atom with an ensemble of harmonic oscillators, each consisting of an electron bound in a quasi-elastic field of force. As a measure for the extension of this field we may take

$$a_s \sim \frac{u_s}{\omega_s}, \tag{3.1.6}$$

where ω_s is the cyclic frequency of the s^{th} electron and where u_s is an "orbital" velocity defined by

$$I_s = \frac{1}{2} \mu u_s^2. \tag{3.1.7}$$

For the most loosely bound electrons in atoms, a_s and u_s are of the same order of magnitude as a_0 and v_0 introduced in § 2.1 while, for the more firmly bound atomic electrons, a_s may be considerably smaller than a_0 and $u_s \gg v_0$.

From the point of view of classical mechanics, the collision problem is particularly simple if $v \gg u_s$ and if, moreover, $b \ll a_s$, where b is the collision diameter defined by (1.1.4). In fact, the duration of a collision with impact parameter p large compared with b will, as regards order of magnitude, be given by $\frac{p}{v}$ and this duration will, therefore, be comparable with $\frac{1}{\omega_s}$ for a value of p given by

$$d_s = \frac{v}{\omega_s}. \tag{3.1.8}$$

For $v \gg u_s$, it is seen that $d_s \gg a_s$ and, for $a_s \gg b$, we thus also have $d_s \gg b$. In this case, we may therefore disregard the influence of the binding forces for $p \ll d_s$, and the statistical distribution of T in such collisions will be given by (3.1.1).

Denoting by i_s the value of p for which $T = I_s$, we get, from (1.1.4), (1.1.10), and (3.1.7),

$$i_s = \left(\frac{1}{2}\right) b \frac{v}{u_s}, \tag{3.1.9}$$

holding for $v \gg u_s$ and $a_s \gg b$, in which case $b \ll i_s \ll d_s$. Under such circumstances, the effect on the atom of an impact with $p \gtrsim d_s$ will amount to only a small perturbation and, since $d_s \gg a_s$, the force exerted by the incident particle will, moreover, be practically uniform over the atomic region in question. The mechanism of the energy transfer in these collisions, thus, presents a problem closely analogous to that of dispersion and absorption of electromagnetic radiation with wave-length large compared with atomic dimensions.

In the case $v \gg u_s$ and $a_s \gg b$, it is thus possible to divide the collisions into two groups which each in its way presents special simplifications and, due to the latitude in the separation between the groups, a high accuracy is obtained by means of such a simple analysis. A detailed calculation (BOHR 1913) gave the result that the total energy transfer is given by

$$\overline{\Delta_\varepsilon E} = N \Delta R B_\varepsilon \sum_s \log \left(k \frac{T_m}{D_s} \right), \qquad (3.1.10)$$

where k is a numerical factor equal to 1.261[1] and where D_s is the energy transfer in a free collision with impact parameter d_s. Since $d_s \gg b$, we have

$$\frac{T_m}{D_s} = \left(\frac{2\, d_s}{b} \right)^2 = \left(\left(\frac{1}{2} \right) \cdot \frac{\mu v^3}{z_1 \varepsilon^2 \omega_s} \right)^2 \qquad (3.1.11)$$

from (1.1.4), (1.1.10), and (3.1.8).

If, for some of the atomic electrons, $u_s \gtrsim v$, the problem is of a more complicated character and the collisions cannot,

[1] This number may be expressed in terms of EULER'S constant C as $4e^{-2C}$.

as above, be divided into two simple groups. The contributions of such electrons to the stopping effect will, however, in general be small. In fact, for $u_s \gg v$, in which case, according to (3.1.4) and (3.1.7), we have $I_s \gg T_m$, the binding forces will prevent a removal of the electron from the atom and will give even the closest collisions a practically adiabatic character. In estimating the electronic stopping of high speed particles we may, therefore, to a first approximation, simply confine the summation in (3.1.10) to electrons for which $I_s < T_m$.

An essential correction in the stopping and ionization formulae, however, may have to be introduced if, for some of the electrons with $u_s < v$, the value of a_s is smaller than b. In fact, in this case, i_s will exceed d_s and the latter quantity will no longer represent an effective adiabatic limit. Under such circumstances, the energy loss may be somewhat larger than given by (3.1.10) while, at the same time, the ionization may be considerably smaller than given by (3.1.3). To this problem, which was first raised by Lamb (1940) in a discussion of the penetration of highly charged fission fragments, we shall return in § 3.3 when considering the scope of the theory in greater detail. We may, however, note that the condition $a_s > b$ is always fulfilled for fast particles if their charges are not very large compared with the elementary unit.

In the stopping formula (3.1.10), relativity refinements are not taken into account, but it is easy to extend the calculations to velocities close to that of light. In fact, for the collisions in which the electron may be regarded as free, the energy transfer will, according to the considerations in § 1.1, in case $p \gg b$, be given by (1.1.12) even for $v \sim c$, since the contraction of the field in the ratio $\frac{1}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}$ is just balanced by a corresponding increase in the field intensity. The shortening of the duration of the collision implies, however, that the effective limit of adiabatic cut-off will be larger than (3.1.8) by a factor γ. From a more detailed calculation (BOHR 1915) like that leading to (3.1.10), taking into account not only the component of the force perpendicular to the motion of the incident particle but also the component parallel to this motion, it follows that, apart from the factor γ in d_s, implying a decrease in D_s by a factor γ^2, the logarithmic argument must be multiplied by $e^{-\frac{v^2}{c^2}}$.

It is true that, for $v \sim c$, special considerations are necessary for collisions with $p \gtrsim b$. Partly, however, these collisions are of minor importance for the total energy loss, if only the logarithmic terms in (3.1.10) are large; partly, in the most important case of $v \sim c$, that of fast β-rays, it is necessary, like for problems of nuclear stopping discussed in § 2.5, to distinguish between the average energy loss and the most probable energy loss of which the latter, which does not depend on the very close collisions, is of primary importance for the analysis of the experiments. To such problems of the distribution of the energy losses we shall return in § 3.4.

The mechanism of stopping of a particle passing through matter may be further elucidated by a direct estimate of the electric field which originates from the polarization of the medium and which acts as a kind of brake on the penetrating particle. Such a procedure is illustrated in Fig. 6, where the centre line, C, represents the path of a positively charged heavy particle, the position of which at the instant considered is indicated by the foot of the arrow. The small circles represent the atoms which, for simplicity, are assumed each to contain only a single electron with orbital extension and velocity a and u. It is assumed that the adiabatic limit d is large compared with a, corresponding to $v \gg u$, and it is also supposed that $a > b$ and that, consequently, d is larger than the ionization limit i. In the figure are indicated the orbits of the electrons prior to the instant considered, although, in the case of ionization, the electron orbits are not traced beyond the particle path. Outside the

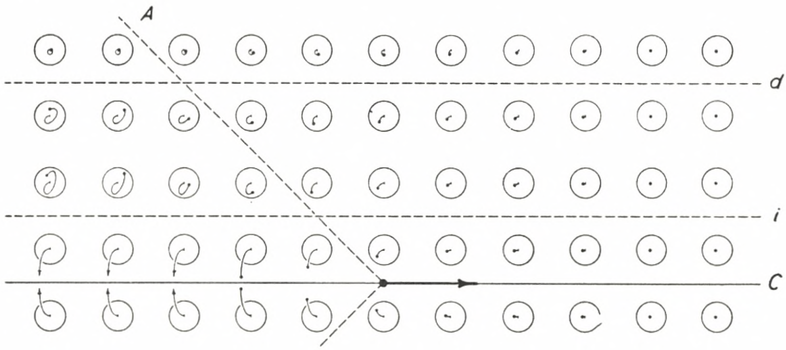

Fig. 6.

adiabatic limit, the positions of the electrons will exhibit complete symmetry with respect to a plane through the particle perpendicular to C, and these electrons will, therefore, give rise to no resultant force on the particle. Inside the adiabatic limit, however, there will be a closer approach of the electrons to the line C behind the particle than in front of it, and these electrons will, therefore, create an electric field directed against the motion of the particle.

To estimate the strength of the field, we may simply calculate the electric charge accumulated in the "wake" of the particle, represented by the cone A containing the atoms for which the collision is practically completed. Since a measure of the displacement of a free electron in a collision for which $p \gg b$, according to the considerations in § 1.1, is given by b, the surplus charge in a section of the cone at distance x behind the particle and of thickness dx will be roughly $2\pi\varepsilon N b x dx$, where, as above, N denotes the number of atoms per unit volume. For the attractive forces of this charge on the particle we, thus, have approximately

$$dF \approx 2\pi z_1 \varepsilon^2 b N \frac{dx}{x}, \qquad (3.1.12)$$

holding for $x \gg b$. For smaller values of x, the displacement perpendicular to the path of the particle will be smaller than b and, as a rough estimate of the total force acting on the particle, we may write

$$F \approx \int_{b/2}^{d} dF \approx 2\pi z_1 \varepsilon^2 b N \log \frac{2d}{b} \qquad (3.1.13)$$

which, as seen from (1.1.4), (3.1.2), and (3.1.11), closely coincides with (3.1.10). A more accurate analysis of the stopping problem from the point of view here indicated is given by A. Bohr (1948).

So far we have assumed that the individual collisions between the particle and the atomic electrons can be considered independently. Like in well-known optical problems we must, however, be prepared that the polarization of the medium, under certain circumstances, may essentially affect the field exerted on the electrons during the passage of the particle. This effect, to which attention was first drawn by Swann (1938), has been considered in detail by Fermi (1940), who showed that, whereas for $v \sim c$ it may imply an essential reduction of the stopping power, the correction is for $v \ll c$ in general very small. Still, as pointed out by Kramers (1947), we meet with an interesting exception for substances like metals containing free electrons. In fact, in this case, corresponding to $\omega_s = 0$, the adiabatic limit (3.1.8) must be replaced by the distance at which the polarization effectively screens the field of the particle. In the treatments of Fermi and Kramers, the influence of the polarization is estimated by considering the substance as a dielectric continuum, but it is of interest that, as shown by A. Bohr (1948), the phenomenon may also be treated from the microscopic point of view, whereby in particular the relationships to ordinary penetration theory are brought out more clearly. The relativity effects are here taken into account by considering the retardation of the fields, and a discussion is given of the intimate connection between the polarization effects and the peculiar radiation phenomena observed by Čerenkov (1934). For these phenomena, the theory has been developed by Frank and Tamm (1937) and by Tamm (1939), who have shown that the electrons radiate when their velocity exceeds the phase velocity of light in some spectral region.

§ 3.2. Quantum Theory of Stopping and Ionization.

The simple mechanical considerations outlined in the preceding paragraph account in a general way for the dependence of the ionization and stopping effects on the charge and velocity of α- and β-rays. A closer test of the ionization formula was for a time prevented by the experimental difficulties of distinguishing between primary ionization and secondary ionization effects produced by fast electrons expelled from the atoms in the more violent collisions, but later it has been definitely established that (3.1.3) gives values for the number of primary ionization processes which, in many cases, are several times too small. Also the stopping formula (3.1.10) was found not to be in full agreement with the measurements on α- and β-rays and, especially in the latter case, to give values for the stopping power appreciably larger than those observed.

In view of the subsequent development of the quantum theory of atomic constitution, it was natural to look for the origin of the deficiencies of the classical formulae in the failure of ordinary mechanics in accounting for atomic reactions and, in particular, in the fact that energy transfers in such reactions can only take place in discrete amounts, unless an electron is completely removed from the atom. Since the excitation energies are never much smaller than the ionization potential, it was, as a first attempt to modify the stopping formula (3.1.10), suggested (Henderson 1922) simply to replace the lower limit by I_s and that, thus, expression (3.1.5) should closely represent the total energy loss. This formula, however, was found (Fowler 1923) to give a stopping power for α-rays only about half the experimental value.

From simple correspondence arguments, it was also soon realized that any such procedure of modifying (3.1.10) was not justifiable (Bohr 1925). In fact, the situation is closely analogous to optical dispersion problems where the average absorption of radiation by virtual atomic oscillators of given frequency and strength is exactly the same in quantum theory as in classical electrodynamics, despite the circumstance that, according to classical theory, any amount of energy can be transferred from the radiation field to the atomic oscillators while, in quantum theory, such transfer can only take place through absorption of individual light quanta.

It is true that the paradoxes here involved, which for a time even led to doubts regarding detailed energy balancing, were felt especially acute in considering collisions the duration of which is short compared with the electronic periods. In fact, in such collisions, the interaction between the particle and the electron would, on ordinary mechanical ideas, have come to an end before the atomic field responsible for the fixation of the stationary energy levels had opportunity to exert any influence on the course of the collisions. After the establishment by Heisenberg of the principle of indeterminacy, and after the recognition that the well-defined application of conservation theorems is complementary to the use of space-time pictures, all such apparent contradictions do, however, disappear (Bohr 1928). In particular, it is impossible by any conceivable experimental arrangement to follow the temporary course of a collision within time intervals smaller than the atomic periods without an uncontrollable amount of energy larger than the spacing between the atomic levels being exchanged between the particle and the measuring apparatus.

The analogy between collision and dispersion phenomena was stressed in a very suggestive way by Fermi (1924), who proposed to estimate the atomic stopping effects for fast particles on the basis of empirical evidence regarding the absorption of high frequency radiation. To this purpose, the perturbing force exerted by the particle on the atom was analyzed, as a function of time, into harmonic components, and the effect of each component was compared with the absorption of electromagnetic waves of corresponding frequency. Although such a procedure involves complications as regards close collisions, it is, at any rate in principle, adequate just for the more distant collisions where the effect of the interatomic forces is of essential importance for the energy transfer. In particular, it is evident that, for impact parameters larger than (3.1.8), the collisions will acquire an adiabatic character, since the perturbing field will no longer contain components which can give rise to atomic resonance.

Shortly after the development of proper quantum mechanics, the problem of the energy transfer in distant collisions was treated in detail by Gaunt (1927), whose calculations for hydrogen gave very nearly the same result as the classical mechanical considerations involved in the deduction of (3.1.10). In fact, the only alteration required by quantum mechanics consists in replacing the summation over the individual atomic electrons with specified oscillation periods by a sum of terms corresponding to the various virtual oscillators giving rise to atomic resonance. Since the statistics of energy transfers in free collisions is the same in quantum mechanics as in classical mechanics, the conclusion was therefore tempting that the stopping formula (3.1.10), apart from the refinements of virtual oscillators, was substantially correct. As regards the ionization effects, however, the arguments in question imply that the classical formula (3.1.3) needs correction. As especially stressed by Williams (1931), a considerable part of the excitation and ionization effects will, in fact, arise from resonance in distant collisions where, on classical mechanics, the energy transfer would consist of individual contributions small compared with I_s.

In the meantime, however, a great progress was achieved by Bethe (1930) who, by a comprehensive quantum-mechanical calculation based on Born's treatment of collision problems, obtained not only an ionization formula differing from (3.1.3), but also a formula for $\overline{\Delta_\varepsilon E}$ which differs essentially from (3.1.10). With the notation used above, Bethe's formula may be written

$$\overline{\Delta_\varepsilon E} = 2 N \Delta R B_\varepsilon \sum_i f_i \log \frac{\left(\frac{1}{2}\right) 2 \mu v^2}{\hbar \omega_i}, \qquad (3.2.1)$$

where the summation is extended to the various virtual atomic oscillators of strength f_i and frequency ω_i. It is significant that the logarithmic term in (3.2.1), in contrast to (3.1.10), does not depend on the charge of the incident particle and, from (3.1.2), it thus follows that the stopping power is proportional to z_1^2, as is a direct consequence of the approximation method applied. Formula (3.2.1) was found to be in very satisfactory agreement with experiments on stopping of α-rays, in particular in light substances where the atomic constants involved can be evaluated with great accuracy.

For the purpose of a comparison with the previous stopping formulae, BETHE's expression may be somewhat simplified. In fact, neglecting minor coupling effects between the electronic bindings, we may attribute the oscillators to the transition probabilities of the individual atomic electrons. Since the total oscillator strength for each electron is close to unity and since, for the most significant transitions of the s^{th} electron, we have $\hbar\omega_s \sim I_s$, formula (3.2.1) may approximately be written

$$\overline{\Delta_\varepsilon E} = 2\,N\,\Delta R\,B_\varepsilon \sum_s{}' \log{(2)}\frac{T_m}{I_s}, \qquad (3.2.2)$$

which is very nearly twice the energy transfer given by (3.1.5). In comparing (3.2.2) and (3.1.10), it is convenient to introduce the quantity $\varkappa$ defined by (1.3.8) and which, in electronic encounters, according to (2.1.6), may be written

$$\varkappa = 2\,\frac{z_1\varepsilon^2}{\hbar\,v} = 2\,z_1\frac{v_0}{v}. \qquad (3.2.3)$$

In fact, for $T_m \gg I_s$, which is presupposed in both formulae, we get, by putting $\hbar\,\omega_s = I_s$, from (3.1.4) and (3.1.11),

$$\left(\frac{(2)\,T_m}{I_s}\right)^2 = \frac{T_m}{D_s}\cdot\varkappa^2. \qquad (3.2.4)$$

We therefore see that, apart from the factor k which may be omitted in this approximation, the two formulae just coincide for $\varkappa = 1$, while (3.2.1) gives smaller or larger values than (3.1.10) according as $\varkappa < 1$ or $\varkappa > 1$.

As regards the relationship between the two formulae, it should be stressed that it is not simply the question of replacing (3.1.10) by (3.2.1) in all applications, but that each formula has its restricted region of validity. On the one hand, $\varkappa \ll 1$ is the necessary and sufficient condition for the applicability of the quantum-mechanical approximation method applied in the deduction of (3.2.1); on the other hand, $\varkappa \gg 1$ is, as we have seen in § 1.3, just the condition for the applicability of orbital pictures in accounting for the collision between two point charges and, in this case, the considerations in § 3.1, supplemented with simple correspondence arguments, may therefore be expected to be appropriate. For fast α- and β-rays, it is seen from (3.2.3) that, in general, $\varkappa$ will be small compared with unity, but the classical formula has acquired renewed interest in the study of the penetration of the highly charged fission fragments for which $\varkappa$ is very large (Bohr 1940 and 1941). For such highly charged particles, however, special considerations are, as already mentioned in § 3.1, necessary since, for some of the atomic electrons with $u_s < v$, we may have $a_s < b$, in which case ionization occurs beyond the limit of adiabatic cut-off (Lamb 1940).

Relativistic treatments of the stopping of fast electrons, for which always $\varkappa \ll 1$, have been given by Bethe (1932) and especially by Møller (1932). The corrections to formula (3.2.1) were found to be just the same as those to be introduced into (3.1.10) for $v \sim c$. This result is intimately connected with the fact that the relativistic modification of the classical formula consists in the addition of terms proportional to z_1^2 in conformity with all effects derived by means of the Born approximation. Moreover, the special quantum-mechanical features involving spin forces and exchange effects, which are of importance for the very violent collisions, may here be disregarded, since they do not contribute to the most probable energy loss (cf. § 3.4).

Since, for $\varkappa \sim 1$, we are outside the regions for the legitimate use of any of the approximation methods applied, the circumstance that (3.2.1) and (3.1.10) coincide so closely for $\varkappa = 1$ must, of course, in some way be regarded as accidental. It was, therefore, of great importance for the scope of the theory that Bloch (1933) succeeded in developing a treatment of the stopping problem which led to a comprehensive formula for $\overline{\varDelta_\varepsilon E}$ applicable for all values of $\varkappa$, provided only that $v > u_s$ and $b < a_s$. In the limits of small values of $\varkappa$, Bloch's formula asymptotically approaches (3.2.1) while, for $\varkappa \gg 1$, it practically coincides with (3.1.10). For intermediate values of $\varkappa$, it never differs appreciably from the one or the other of these two formulae which, therefore, are proved to supplement each other with high approximation. This result is of special value because, in many of the most important applications, $\varkappa$ will be neither very large nor very small and, consequently, none of the previous deductions could claim great accuracy.

Bloch's treatment rests upon a generalized impact parameter method in which the collisions are specified by the distance of the path of the particle from the nucleus of the atom. As regards the interaction of the particle with an electron bound in the atom, this procedure which reduces the collision to a one-body problem is, in fact, equivalent to the ordinary quantum-mechanical treatment in configuration space, if only the transfer of momentum is small compared with the total momentum of the incident particle (Mott (1931); cf. also A. Bohr (1948)). This specification of the encounters makes it possible, like in the classical treatment, to separate between distant collisions where the perturbing force is uniform over the atom, and closer collisions the duration of which is short compared with atomic periods.

While the distant collisions may, at any rate if $b < a_s$, be treated as a simple dispersion problem it is permissible, as shown by Bloch, to disregard the atomic binding forces in estimating the average energy transfer in close collisions. This circumstance allows an evaluation of $\overline{\Delta_\varepsilon E}$ without any detailed examination of the statistical distribution of the energy losses in individual collisions like that implied in the deduction of (3.1.10) and (3.2.1). Such an analysis is, however, required for the discussion of various other penetration problems, in particular of the ionization effects.

§ 3.3. Statistics of Electronic Collisions.

In collisions between fast particles and atomic electrons we meet, as we have seen, with two especially simple types which may be referred to as "free collisions" and "resonance effects", respectively. In the former case, we have essentially to do with a two-body problem while, in the latter case, we are presented with a perturbation problem analogous to that met with in dispersion theory. Although, strictly speaking, we are here dealing with limiting cases, it is possible, as we shall see, to a wide extent to classify the collisions into these two simple types and thereby to obtain a general survey of the statistical distribution of the individual energy transfers and, in particular, of the dependence of this distribution on the charge and velocity of the incident particle.

In order to specify the notations, we note that, if the duration of the encounter is considered as short compared with the atomic periods and if, moreover, the energy transfers to the electron are large compared with the ionization potentials, the influence of the atomic binding forces on the course of the interaction may, with high approximation, be disregarded. Under such circumstances, we may speak of free collisions in which the statistical distribution of the individual energy transfers, T, will be given by (3.1.1). A main problem with which we are concerned will be to examine the region of T for which, under the various circumstances, the conditions for the applicability of this formula are fulfilled.

Simple resonance effects occur if the field exerted by the incident particle is practically uniform over the atomic region and if, furthermore, its intensity is so weak that the probability of atomic excitation or ionization in each encounter is small. Under these circumstances, the statistical distribution of the individual quantum-mechanical transition processes can always be obtained by the perturbation method applied by BETHE, and his results may, therefore, be suitably adapted to the various possible cases including those for which such a treatment is not valid for the penetration problem as a whole. From the considerations in § 3.2 it follows, moreover, that, in this type of collision, the average energy transfer determining the stopping effect is just the same as that obtained by a classical calculation in which the atom is treated as an ensemble of harmonic oscillators.

In order not to complicate the arguments, we shall in the first instance disregard finer details of the atomic structure and simply characterize the binding of the s^{th} electron by a cyclic frequency ω_s, related to the ionization potential I_s by

$$I_s = \hbar\,\omega_s\,, \tag{3.3.1}$$

and by a length a_s defined by

$$a_s = \frac{\hbar}{\mu\,u_s}\,, \tag{3.3.2}$$

where u_s represents the "orbital" velocity given by (3.1.7). While $\mu\,u_s$ is of the same order of magnitude as the quantum-mechanical expectation value for the momentum of the electron in its undisturbed state, a_s represents the accuracy with which the electron may be localized without the uncertainty of its energy exceeding I_s and will, for bindings with effective quantum numbers comparable with unity, be a measure of the radial extension of the orbital region.

As already indicated in the preceding, the problem of the electronic collision effects produced by fast particles depends essentially on the value of the quantity $\varkappa$ given by (3.2.3), and it will be convenient to treat the cases $\varkappa > 1$ and $\varkappa < 1$ separately in order to bring out as clearly as possible the differences as well as the conformities of the penetration problem in these two cases.

For $\varkappa > 1$, the individual encounters between the incident particle and a free electron can be approximately described in terms of orbital pictures and, to the purpose of classifying the collisions, we may apply the considerations in § 3.1. For the impact parameter i_s corresponding to a free collision with $T = I_s$, we have from (3.1.9), by means of (1.1.4), (3.2.3), and (3.3.2),

$$i_s = \varkappa\,a_s\,, \tag{3.3.3}$$

holding for $v \gg u_s$. For the adiabatic limit d_s defined by (3.1.8), we have, from (3.1.7), (3.3.1), and (3.3.2),

$$d_s = \frac{v}{\omega_s} = \eta_s a_s, \qquad (3.3.4)$$

where

$$\eta_s = \frac{2v}{u_s} \qquad (3,3,5)$$

is a convenient abbreviation, Furthermore we have

$$\frac{d_s}{i_s} = 2\frac{a_s}{b} = \eta_s \cdot \varkappa^{-1}, \qquad (3.3.6)$$

as is seen from (3.1.9) and (3.3.3)[1].

For $\varkappa < \eta_s$ it follows from (3.3.6) that $i_s < d_s$, in which case the collisions with $p < i_s$ may be classified as "free", since $T > I_s$ and since the duration of the interaction is short compared with $\frac{1}{\omega_s}$. According to (3.3.3) we have, moreover, $i_s > a_s$, and encounters for which $p > i_s$ are, therefore, of the simple resonance type. To a first approximation the contribution of the s^{th} electron to the stopping will thus be given by the sum of two terms,

$$(\overline{\Delta_s E})_f = N \Delta R\, B_\varepsilon \log \frac{T_m}{I_s} \qquad (3.3.7)$$

and

$$(\overline{\Delta_s E})_r = N \Delta R\, B_\varepsilon \log \frac{I_s}{D_s}, \qquad (3.3.8)$$

representing the energy losses due to free collisions and resonance effects, respectively.

[1] In the formulae with validity restricted to $\varkappa > 1$, we omit the extra numerical factors to be introduced if the incident particles are electrons since, in this case, for $v \gg v_0$, the value of $\varkappa$ is always small compared with unity.

The energy transfer in free collisions, thus, corresponds to (3.1.5), while the additional contribution to (3.1.10) is to be ascribed to resonance effects. Expressing the logarithmic arguments in terms of η_s and $\varkappa$, one finds by means of (3.1.4), (3.1.7), (3.2.4), and (3.3.5)

$$\frac{T_m}{I_s} = \eta_s^2 \quad \text{and} \quad \frac{I_s}{D_s} = \eta_s^2 \varkappa^{-2}, \tag{3.3.9}$$

from which, in particular, it follows that $(\overline{\Delta_s E})_f$ is larger than $(\overline{\Delta_s E})_r$ for $\varkappa > 1$. With increasing values of $\varkappa$, the resonance effects give a decreasing contribution which, for the s^{th} electron, vanishes for $\varkappa = \eta_s$.

If $\varkappa > \eta_s$, in which case $i_s > d_s$, the values of p for which the collisions may be regarded as "free" no longer extend to i_s, since for $p > d_s$ the duration of the encounter exceeds $\frac{1}{\omega_s}$. On the other hand, all interactions with $p > d_s$ will not be of a purely adiabatic character since, during the collision, the binding of the electron may be disrupted. This is also directly indicated by the circumstance that, for $\varkappa > \eta_s$, the displacement of the electron which during a free collision is of the order of b is no longer small compared with atomic dimensions, but may exceed a_s, as is seen from (3.3.6).

In order to estimate the limiting value d_s^* of p, for which the probability of ionization during the passage of the particle is still of the order of unity, we may, as effective duration for free energy transfer for $p > d_s$, instead of $\frac{p}{v}$, take $\frac{1}{\omega_s}$. By a simple calculation, one thus obtains

$$(d_s^*)^2 = i \cdot d_s, \tag{3.3.10}$$

holding for $v > u_s$ and $d_s^* > b$. For $i_s = d_s$, this expression gives $d_s^* = d_s$, and the two regions $\varkappa < \eta_s$ and $\varkappa > \eta_s$ are thus joined smoothly together. The ionization processes outside d_s may be compared with a so-called cold emission by which an electron is pulled out of an atom by a static field of force (Lamb 1940). Due to the fact that d_s^*, as follows from (1.1.4), (3.1.9), and (3.3.4), is independent of v, such considerations give essentially the same results as the above estimate based on arguments of dynamics.

Since, for $p < d_s^*$, the ionization process will take place at a comparatively early stage of the encounter, the total energy transfer will be practically the same as if the electron were free. Denoting by D_s^* the energy transfer in a free collision with $p = d_s^*$, we therefore have, to a first approximation,

$$\overline{\Delta_s E} = (\overline{\Delta_s E})_f = N \Delta R B_\varepsilon \log \frac{T_m}{D_s^*}, \qquad (3.3.11)$$

where, according to (1.1.10), (3.1.9), (3.3.6), and (3.3.10),

$$\frac{T_m}{D_s^*} = \eta_s^3 \varkappa^{-1}, \qquad (3.3.12)$$

holding for $T_m \gg D_s^*$ or $\varkappa \ll \eta_s^3$. For still larger values of $\varkappa$, where $d_s^* \gtrsim b$, the whole problem is more intricate, since the collisions will have a large probability of leading to capture of the electron by the particle. Actually, the continual capture and loss of electrons along the path, a problem which we shall consider more closely in Chapter 4, will in such cases be determining for the penetration effects.

For $\varkappa < 1$, classical mechanical ideas, such as orbits of the particles during the encounter, fail completely in accounting for the individual collision effects. As mentioned in § 3.2, the collisions may, however, still be specified by a generalized impact parameter P equal to the distance of the path of the incident particle from the nucleus of the atom and definable with a latitude small compared with atomic dimensions. On this basis, we shall see that, also for $\varkappa < 1$, a separation between "free" collisions and "resonance" effects is possible in first approximation and that the two cases correspond to collisions where the particle passes through and outside the atomic region or to $P < a_s$ and $P > a_s$, respectively.

In the case of close collisions between the incident particle and an atomic electron, the problem is in essential respects analogous to the scattering problem considered in § 1.4. Indeed, for $v \gg u_s$, the initial stage of the collision may, in relative coordinates, be described by a wave function which, over time intervals comparable with the duration of the penetration of the particle through the atom, closely resembles that of a wave-packet of radial extension a_s and moving with a well-defined velocity v. The limited size of the wave-packet will obviously have a similar influence on the scattering as a screening of the field of force between the colliding particles, of the type given by (1.4.1), for $a \sim a_s$. Since $\lambda\!\!\!^{-} \ll a_s$, according to (1.3.2) and (3.3.2), the angular scattering distribution in relative coordinates will, thus, be of the type $R''_a(\vartheta)$, corresponding to the Rutherford law for angles larger than ϑ''_a given by (1.4.8).

For the energy transfer we should, therefore, expect a statistical distribution given by (3.1.1) for $T > A''_s$, where, according to (1.1.8),

$$A''_s = T_m \left(\frac{\vartheta''_{a_s}}{2}\right)^2 = I_s, \qquad (3.3.13)$$

as follows from (3.1.4) and (3.3.2)[1]. Since energy transfers smaller than I_s would, thus, be very improbable, it follows that the atomic binding forces can have no essential influence on the distribution of T, and the collisions may, consequently, be classified as "free". The contribution to the stopping effect of collisions with $P < a_s$, therefore, approximately amounts to

$$(\overline{\Delta_s E})_f = N \Delta R B_\varepsilon \log \frac{T_m}{I_s}, \qquad (3.3.14)$$

an estimate identical with (3.3.7), holding for $1 < \varkappa < \eta_s$.

In the distant collisions where $P > a_s$, the force exerted by the particle will be approximately constant over the atomic region. Moreover, we are in such collisions dealing with a perturbation problem, corresponding to the fact that, for $\varkappa < 1$, we have $i_s < a_s$, according to (3.3.3), and that, therefore, a classical calculation would lead to energy transfers smaller than I_s. In quantum mechanics we have thus to do with typical "resonance" effects which will give rise to an average energy transfer equal to that to be expected from a classical interaction between the particle and the atomic oscillators. The contribution to the stopping effect of the collisions with $P > a_s$ will, thus, be

$$(\overline{\Delta_s E})_r = N \Delta R B_\varepsilon \log \frac{A'_s}{D_s}, \qquad (3.3.15)$$

where A'_s is the energy transfer calculated in classical mechanics for $p = a_s$.

1) It may be noted that (3.3.13), and consequently (3.3.14), holds also if the incident particle is an electron, in which case the value of ϑ''_{a_s}, depending on the reduced mass, is doubled, while T_m is four times smaller.

From (3.3.13) and from (1.4.3), and (1.4.8), we have

$$A'_s = I_s \varkappa^2 \tag{3.3.16}$$

and by means of (3.2.4) we, thus, find

$$(\overline{\Delta_s E})_r = N \Delta R B_\varepsilon \log \frac{(4)\, T_m}{I_s}, \tag{3.3.17}$$

where, as usual, the bracketed italic factor in the logarithmic argument refers to the case of electrons as incident particles.

It will be seen that the sum of (3.3.14) and (3.3.17) just corresponds to Bethe's expression for $\overline{\Delta_\varepsilon E}$ in the simplified form (3.2.2). While, as mentioned above, $(\overline{\Delta_s E})_r$ is always smaller than $(\overline{\Delta_s E})_f$ for $\varkappa > 1$, we see that for $\varkappa < 1$ the two contributions are essentially equal, in accordance with the circumstance, already referred to in § 3.2, that Henderson's formula, in which the resonance effects were not taken into account, gave an energy loss only about half that obtained by Bethe.

For the illustration of the considerations in this paragraph, a survey of the characteristic features of the statistical distribution of the individual energy losses for different values of $\varkappa$ is given in Fig. 7. The cases marked I, II, and III correspond to different values of the charge of the incident particle while, for the sake of comparison, the velocity v as well as the strength of the electron binding characterized by I_s are taken to be the same in all three cases. Like in the instructive diagrams used in similar problems by Williams (1931), we have chosen as abscissa $\log \frac{T_m}{T}$ and as ordinate the ratio ξ between the actual differential cross-section for energy transfers T and the equation (3.1.1), corresponding to collisions with a free electron. This choice of coordinates

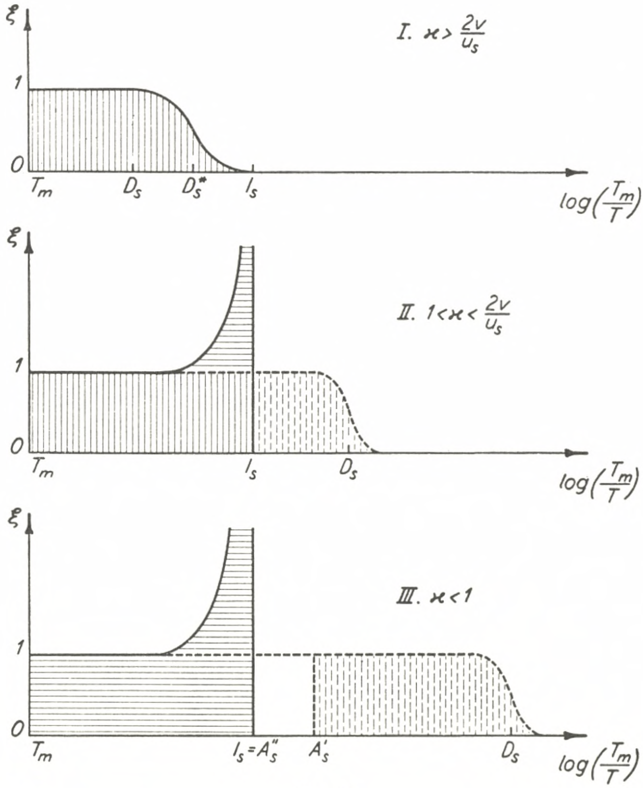

Fig. 7.

implies in particular that, as follows from (2.3.4), the contribution to $\overline{\Delta_s E}$ from collisions for which $T_1 < T < T_2$ is simply proportional to the area between the curve and the axis of abscissa, limited by the ordinate lines corresponding to T_1 and T_2.

The diagrams I and II refer to problems for which $\varkappa > 1$. In I, the value of $\varkappa$ is chosen larger than $\eta_s = \dfrac{2v}{u_s}$ and, thus, $D_s > I_s$, in which case all the collision effects can, at any rate approximately, be treated on classical mechanics.

This is indicated by the vertical hatching of the whole area below the curve extending from T_m essentially to D_s^*. In the case illustrated, $D_s \ll T_m$, and the abscissa corresponding to D_s^* is, therefore, about half way between D_s and I_s, as follows from (3.3.10).

Diagram II represents the case $1 < \varkappa < \frac{2v}{u_s}$, where $I_s > D_s$. The full-drawn curve again corresponds to the actual distribution of T with the slight simplification that, in order not to complicate the figure, no distinction is made between discrete resonance lines giving rise to atomic excitation and ionization processes with $T \sim I_s$. The area below the curve is separated into two regions corresponding to the distinction between free collisions and resonance effects. As indicated by the vertical hatchings, the former region may be accounted for by means of classical mechanics, while the horizontal hatchings indicate that the resonance effects are proper quantum phenomena. The broken curve represents the distribution to be expected if the energy transfers in all the individual collisions could be calculated by means of classical mechanics. Since, as mentioned above, such a calculation gives the correct result for the total energy loss, the area characterized by broken vertical hatchings will be just equal to the area representing the total energy loss in resonance effects and indicated by horizontal hatchings although in part falling outside the figure.

Diagram III illustrates the case $\varkappa < 1$. Here, the whole area below the full curve will, as indicated by the horizontal hatchings, correspond to effects which defy any unambiguous use of classical pictures. The broken curve gives the distribution to be expected if classical mechanics could be applied. In constrast to the case $\varkappa > 1$, the area beyond $T = I_s$ represents an energy loss which is greater than the contribution of the resonance effects, in accordance with the failure of the classical mechanical formula (3.1.10) for $\varkappa < 1$. As appears from the above discussion, it is, in fact, only the part of the area corresponding to $D_s < T < A'_s$ and indicated by broken hatchings which, according to dispersion theory, is equivalent to the contribution due to resonance effects. The area corresponding to $A'_s < T < A''_s = I_s$ represents the difference in the values for $\overline{\Delta_s E}$ given by formulae (3.1.10) and (3.2.1).

The diagrams in Fig. 7 clearly illustrate the relative importance of the resonance effects for the different values of $\varkappa$. While, in III, the contribution of these effects to the total energy loss just equals the contribution of the free collisions, it is shown by II how, for $\varkappa > 1$, the resonance effects may be considerably less important for the stopping. In the extreme case $\varkappa > \frac{2v}{u_s}$ illustrated by I, there is no question of any simple resonance effects.

The considerations in this paragraph which, in the first place, aim at a survey of the distribution of the individual energy transfers in electronic collisions and of their dependence on the value of $\varkappa$, obviously have a somewhat qualitative character. In particular, it need hardly be emphasized that the distinction between free collisions and simple resonance effects involves a considerable latitude. The high degree of accuracy in the estimate of the average energy loss obtainable within a large region of $\varkappa$ by the treatment of Bloch depends just on the possibility of avoiding such a distinction. In fact, as mentioned, it is in this treatment only necessary to separate between such distant collisions, where the analogy to the simple dispersion phenomena is complete, and the closer collisions of duration short compared with atomic periods, where the atomic forces, though determining for the distribution of the individual energy losses, have no influence on the average energy transfer. In many other penetration problems, however, the details of the distribution of the individual energy losses are essential and the above analysis, which is more in line with that of Williams (1932), goes essentially beyond the scope of Bloch's treatment.

In particular, the distinction between free collisions and resonance effects allows, at any rate approximately, an estimate for all values of $\varkappa$ of the number of ions produced along the path. Still, to this purpose, it is, of course, necessary to separate between resonance effects which result in a removal of an electron from the atom and those which produce only atomic excitation. Since, however, for the majority of the resonance effects, the duration of the collisions will be short compared with the atomic periods which are, moreover, for a given electron all of the same order of magnitude, the relative transition probabilities will, to a first approximation, be independent of the charge and velocity of the incident particle. On this basis, we shall in the following paragraphs, by making use of the results of Bethe deduced for $\varkappa \ll 1$, consider the ionization problems for all values of $\varkappa$.

An extension of the above considerations to velocities close to that of light presents no essential difficulties. Since $c = 137 v_0$, it is seen from (3.2.3) that, for $v \sim c$, we have always $\varkappa \lesssim 1$ even for the largest conceivable values of z_1 and, in considerations of the relativistic corrections to electronic stopping formulae, the discussion may, therefore, be confined to such values of $\varkappa$. As pointed out in § 3.1 and § 3.2, the retardation of the field of the incident particle will not appreciably affect the energy transfer in free collisions, but will primarily imply an increase in the adiabatic limit by a factor γ. While, therefore, formula (3.3.14) for the contribution to $\overline{\Delta_s E}$ from free collisions is substantially correct even for $v \sim c$, the contribution from the resonance effects given by (3.3.17) will be somewhat increased. Since an increase in d_s by a factor γ implies a decrease in D_s by a factor γ^2, we have thus

$$(\overline{\Delta_s E})_r = N \Delta R B_\varepsilon \log \left(\frac{(4)\, T_m}{I_s} \gamma^2 \right) \qquad (3.3.18)$$

approximately corresponding to the relativistic correction terms in BETHE's and MØLLER's formulae. As regards the distribution of the individual resonance processes on the various transition possibilities, we may, for the same reasons as given above, use BETHE's calculations also in the relativistic region.

§ 3.4. Penetration of High Speed Particles in Light Substances.

The treatment of electronic encounters indicated in the preceding paragraphs offers a basis for the estimate of stopping and ionization effects of particles with velocities large compared with the "orbital velocities" of the atomic electrons in the materials penetrated. In light substances like hydrogen and helium, where all the electronic velocities are comparable with v_0, the treatment thus covers the case of high speed particles for which $v \gg v_0$. In the present paragraph, we shall see that the various penetration effects may, under such circumstances, be simply analyzed by means of the distinction between free collisions and resonance effects discussed in § 3.3.

As regards the average energy loss, the formulae may be comprehensively written

$$(\overline{\Delta_\varepsilon E})_f = N \Delta R B_\varepsilon \sum_s \log \left\{ \left(\frac{1}{4} \right) \eta_s^2 \left[\frac{\varkappa}{\eta_s} \right]^{-1} \right\} \qquad (3.4.1)$$

and

$$(\overline{\Delta_\varepsilon E})_r = N \Delta R B_\varepsilon \sum_s{}' \log \{\eta_s^2 [\varkappa]^{-2}\}, \qquad (3.4.2)$$

where square bracketing indicates that the quantity within the brackets is to be replaced by unity, if smaller than 1, and where the summation is to be extended over the various electrons in the atoms which give a positive value for the logarithm. With this notation the total energy loss may be written

$$\overline{\Delta_\varepsilon E} = N \Delta R B_\varepsilon \sum_s{}' \log \left\{ \left(\frac{1}{4}\right) \eta_s^4 [\varkappa]^{-2} \left[\frac{\varkappa}{\eta_s}\right] \right\}. \qquad (3.4.3)$$

As mentioned in § 3.1 and § 3.2, the value of $\overline{\Delta_\varepsilon E}$ may, within large regions of $\varkappa$, be calculated more accurately without introducing a sharp separation between free collisions and resonance effects, but it may be noted that (3.4.3), for $\varkappa < \eta_s$, practically coincides with Bloch's formula.

For $v \sim c$, where $\varkappa$ is always smaller than unity, it follows from the arguments in § 3.3, that, while $(\overline{\Delta_\varepsilon E})_f$ is not essentially influenced by relativity effects, the contribution of the resonance effects is, according to (3.3.18), increased to

$$(\overline{\Delta_\varepsilon E})_r = N \Delta R B_\varepsilon \sum_s{}' \log \eta_s^2 \gamma^2 , \qquad (3.4.4)$$

corresponding approximately to the formulae of Bethe (1932) and of Møller (1932). For large values of γ, it must, however, be taken into account (Fermi 1940) that the mutual interaction between the atoms (cf. § 3.1) implies a reduction in the resonance effects, whereby the stopping power becomes independent of the binding forces and is determined only by the electron density of the medium.

Like in the case of nuclear stopping, it may be of importance also to consider the statistical distribution of $\Delta_\varepsilon E$. In particular, we shall examine the mean square deviation Ω_ε^2 determining for the so-called range straggling, which will be more closely considered in § 5.4. The first detailed treatments of this phenomenon were given independently by FLAMM (1914 and 1915) and by BOHR (1915) on the basis of simple considerations of classical mechanics. Assuming that the individual energy losses are distributed according to (3.1.1), one gets from (2.3.5), by neglecting the lower limit T_1 in comparison with $T_2 = T_m$ and by summing over the z_2 electrons in the atom,

$$\Omega_\varepsilon^2 = N \Delta R B_\varepsilon T_m z_2, \tag{3.4.5}$$

an expression which is especially simple, being independent of v, according to (3.1.2) and (3.1.4).

Since, also in quantum mechanics, the distribution of the individual energy losses will be given by (3.1.1) for large values of T, formula (3.4.5) must be expected to represent a first approximation. In a more accurate estimate it should be taken into account (WILLIAMS 1932) that the distribution of T is given by (3.1.1) only for $T \gg I_s$, while the collisions which, on classical mechanics, would lead to smaller energy transfers actually give rise to resonance effects in which $T \sim I_s$. This problem has, for $\varkappa < 1$, been examined in detail by LIVINGSTON and BETHE (1937) and the treatment has been extended to all values of $\varkappa$ by ŢIŢEICA (1937) by means of the Bloch method. The corrections to (3.4.5) are, however, small and, since the additional terms are of higher order in η_s^{-1}, they would hardly seem to go much beyond the degree of approximation obtainable by present methods.[1)]

[1)] Note added in proof: It is in this connection of interest that, while earlier determinations of α-ray straggling gave values for Ω somewhat larger than (3.4.5), recent investigations of proton straggling (C. B. MADSEN and P. VENKATESWARLU, Phys. Rev., in press), based on the measurement of the broadening of nuclear resonance curves have given results in close agreement with the simple formula.

In considering the statistical distribution of the energy losses $\Delta_\varepsilon E$, we may proceed in quite the same way as in the discussion of nuclear stopping effects in § 2.4. The only alteration in the corresponding formulae consists, in fact, in the replacement of the single logarithmic term in $\overline{\Delta_\nu E}$ by the sum of such terms in $\overline{\Delta_\varepsilon E}$. In the case of heavy incident particles ($m_1 \gg m_2 = \mu$), we may, therefore, conclude that, for not too small fractions of the range, the energy losses in electronic collisions will be distributed according to a simple Gaussian law with half width Ω_ε. If, however, the incident particles are electrons, the distribution will be of a more complicated character and will consist essentially of an approximately Gaussian peak and a tail extending far beyond the width of the peak. Introducing, in complete analogy to the considerations in § 2.4,

$$T^* = N\Delta R B_\varepsilon z_2 \tag{3.4.6}$$

one finds

$$\overline{\Delta_\varepsilon^* E} = N\Delta R B_\varepsilon \sum_s \log\left\{\frac{1}{4}\eta_s^4 \frac{T^*}{T_m}\right\} \tag{3.4.7}$$

and

$$(\Omega_\varepsilon^*)^2 = N\Delta R B_\varepsilon z_2 T^* \tag{3.4.8}$$

for the most probable energy loss $\overline{\Delta_\varepsilon^* E}$ and the width Ω_ε^* of the Gaussian peak.

A simple survey of the ionization phenomenon may likewise be obtained by separating between free collisions and resonance effects. The contribution of free collisions which, according to their definition, all lead to ionization is given by the Thomson formula (3.1.3), provided $\varkappa < \eta_s$. For larger values of $\varkappa$, the number of free collisions is obtained by integrating $N\Delta R d\sigma$ from D_s^* to T_m. Neglecting higher order terms, we thus have for the number of ions

$$(\omega_I)_f = N\varDelta RB_\varepsilon \sum_s \frac{1}{I_s}\left[\frac{\varkappa}{\eta_s}\right]^{-1}. \qquad (3.4.9)$$

As regards the resonance effects, it follows from the narrow distribution of T around I_s, as indicated in Fig. 7, that the combined number of ionization and excitation processes involving the s^{th} electron will always be close to $(\overline{\varDelta_s E})_r$ divided by I_s. For the number of ions, we may thus write

$$(\omega_I)_r = N\varDelta RB_\varepsilon \sum_s \delta_s \cdot \frac{1}{I_s} \log\left(\eta_s^2 [\varkappa]^{-2}\right), \qquad (3.4.10)$$

where δ_s is a numerical factor. The value of δ_s depends on the distribution of the resonance effects over the excitation and ionization states and is, therefore, as follows from the arguments in § 3.3, to a first approximation independent of the charge and velocity of the particle and, in particular, of $\varkappa$.

For hydrogen atoms, Bethe (1930) has calculated δ to be 0.28 and has also attempted an estimate of the ionization for atoms containing several electrons, with special reference to the influence of their mutual coupling. Since, however, the deviations from the Coulomb field due to the electronic screening are neglected, these latter estimates are only of a cursory character. This point has been recently stressed by Fano (1946), who has shown that δ_s may depend very essentially on the screening. In particular, Fano finds that, for the most loosely bound electrons in the atom, which contribute primarily to the ionization, the variation in δ_s will approximately compensate the considerable differences in I_s. Thus, a basis was obtained for an explanation of the remarkable similarity of the ionization phenomena in different substances of widely different minimum ionization potentials.

For the total number of collisions leading to ionization we get, from (3.4.9) and (3.4.10),

$$\omega_I = N\varDelta RB_\varepsilon \sum_s{}' \frac{1}{I_s}\left\{\left[\frac{\varkappa}{\eta_s}\right]^{-1} + 2\,\delta_s \log\left(\eta_s\,[\varkappa]^{-1}\right)\right\}. \qquad (3.4.11)$$

It is of interest to point out that while, due to the contribution of the resonance effects, the ionization may, for $\varkappa \ll \eta$, be several times larger than given by the Thomson formula (3.1.3), this latter formula should be very nearly correct for $\varkappa \sim \eta$ which, as we shall see, corresponds to the case of fission fragments over a considerable part of their range. For $\varkappa \gg \eta$, the ionization should be essentially less than given by (3.1.3).

For $v \sim c$, where always $\varkappa < 1$, it follows from the arguments in § 3.3 that the only correction to (3.4.11) arises from the increase in $(\overline{\varDelta_\varepsilon E})_r$ as given by (3.3.18). We thus have

$$\omega_I = N\varDelta RB_\varepsilon \sum_s{}' \frac{1}{I_s}\left(1 + 2\delta_s \log \eta_s \gamma\right) \qquad (3.4.12)$$

which, for hydrogen, corresponds to the expression derived by Møller (1932).

As regards comparison of the ionization formulae with experiments, it is usually impossible directly to discriminate between the primary ionization consisting in the expulsion of electrons in collisions with the incident particle, and the secondary ionization produced by the electrons expelled with an energy greater than the lowest ionization potential of the substance. Electrons with such energies practically only originate from the free collisions but, of course, in the ionization they produce, resonance effects as well as free collisions must be taken into account. An accurate estimate of the secondary ionization presents a very complicated phenomenon which can be treated only approximately on present theories, since a considerable part of the effect will be due to collisions between atoms and electrons with velocities of the same order of magnitude as v_0.

An analysis of the phenomenon has been attempted by BETHE (1930), WILLIAMS (1932), BAGGE (1937), and especially by FANO (1946), who found it possible to account for the experimental result that the total ionization corresponds to an energy expenditure per ion between 30 and 40 electron volts, approximately independent of the substance. This quantity must also be expected to vary only very slowly with the charge and velocity of the incident particle, although it may depend somewhat on the relative importance of free collisions and resonance effects for the primary ionization. In particular, it must be noted that the average energy expenditure per ion may not be quite the same for fission fragments as for fast α-particles, due to the much smaller contribution of resonance effects in the former case. Some indication that the differences are only comparatively small has, however, been obtained by LASSEN (1946), who has shown that the ratio between the ionization per energy loss in different substances is closely the same for fission fragments as for α-particles.

§ 3.5. Penetration of High Speed Particles in Heavy Substances.

The problem of the interaction between swift particles and heavy atoms is of a more intricate character than the penetration problems in light substances discussed in the preceding paragraph. Not only would a detailed analysis of the atomic oscillators be highly complicated, but even for $v \gg v_0$ the orbital velocities of the most firmly bound electrons may be comparable with or exceed v, in which case the simple theory discussed in the preceding no longer applies. In case of $\varkappa < 1$, a thorough discussion of the available experimental material has been given by Livingston and Bethe (1937), who developed semi-empirical formulae and, in this connection, have also given a theoretical treatment of the contribution to the stopping power of atomic electrons with $u_s \gtrsim v$.

An interesting attempt has been made by Bloch (1933a) to obtain a comprehensive stopping formula by comparing the atom, in the distant collisions where the atomic constants are of importance, with the simplified model of a Thomas-Fermi gas. While this procedure may be appropriate for velocities exceeding the largest values of u_s, it is more difficult to apply to smaller velocities. For the present considerations, which primarily aim at bringing out the dependence of the penetration effects on the charge and velocity of the particle, we shall, therefore, attempt, on the basis of the formulae of the preceding paragraphs and by means of a simple model of the atom, to give approximate expressions for the stopping power of heavy atoms, covering all values of $\varkappa$. In particular, such estimates should be suited for the comparison between the stopping effects of fast α-rays for which $\varkappa < 1$ and of fission fragments for which $\varkappa$ is essentially larger than unity.

In the estimate of the distribution of the electronic velocities u_s which enter into the formulae of § 3.4, it is convenient to write

$$u_s = \frac{z_s^*}{\nu_s} v_0, \tag{3.5.1}$$

where z_s^* is a measure of the strength of the field in the region in which the electron is bound, as compared with the field of a hydrogen nucleus, and where ν_s is the so-called effective quantum number. In the case of heavy atoms, the most firmly bound electrons belonging to the shells K, L, etc. move in a field which is approximately Coulombian with $z_s^* \sim z_2$ and have values of ν_s very nearly equal to 1, 2, etc., respectively; furthermore, the most loosely bound electrons for which $z_s^* \sim 1$ again have values of ν_s of the same order as unity. Over a large intermediate region, however, ν_s will have a flat maximum corresponding to values close to $z_2^{1/3}$, a result which is in conformity with the analysis of the electron binding by the Thomas-Fermi statistical method.

Since z_s^* approximately represents the number $n(u_s)$ of atomic electrons with velocities smaller than u_s, we have thus

$$n(u_s) \sim z_2^{1/3} \frac{u_s}{v_0}, \tag{3.5.2}$$

holding for $v_0 < u_s < z_2^{2/3} v_0$. For values of u_s outside this region, we must for n use some function of z_2 and u_s which, for $u_s \sim v_0$, is of the order of unity, while it approaches z_2 for $u_s \sim z_2 v_0$. In order, however, to avoid complications, we shall in the present survey confine ourselves to the simple expression (3.5.2) and postpone the question of its limitations to the discussion of special problems in the following chapters.

As regards the stopping effect, we may first consider the more simple expression for the contribution of resonance effects. Thus, from (3.3.5) and (3.4.2), we get

$$(\overline{\Delta_\varepsilon E})_r = 2N\Delta R\, B_\varepsilon \int \log\left\{\frac{2v}{u_s}[\varkappa]^{-1}\right\} dn(u_s). \tag{3.5.3}$$

In evaluating this integral we may, in first approximation, use the simple expression (3.5.2) and integrate from $u_s = 0$

to the value of u_s for which the logarithm vanishes. For the more firmly bound electrons, for which the respective terms in (3.5.3) no longer apply, the contribution to the stopping power will, for heavy atoms, be of only minor importance, provided $v \gg v_0$. We thus get

$$(\overline{\Delta_\varepsilon E})_r = 2\, N \Delta R B_\varepsilon n_\varepsilon\, [\varkappa]^{-1}, \tag{3.5.4}$$

where

$$n_\varepsilon = z_2^{1/3} \cdot \frac{2\,v}{v_0}, \tag{3.5.5}$$

according to (3.5.2), may be regarded as a measure of the number of atomic electrons effectively involved in the stopping phenomenon.

The contribution of free collisions to the stopping effect may be estimated from (3.4.1). Confining ourselves to the case of penetrating particles of mass large compared with that of the electron, one finds by integrating, for $\varkappa < 1$, from $u_s = 0$ to $u_s = 2\,v$, while, for $\varkappa > 1$, the integration has to be performed in two parts, from 0 to $2\,v\varkappa^{-1}$ and from $2\,v\varkappa^{-1}$ to $2\,v\varkappa^{-1/3}$ (cf. p. 85),

$$(\overline{\Delta_\varepsilon E})_f = N \Delta R B_\varepsilon n_\varepsilon \left(3\,[\varkappa]^{-1/3} - [\varkappa]^{-1}\right) \tag{3.5.6}$$

and, by adding (3.5.4), one obtains

$$(\overline{\Delta_\varepsilon E}) = N \Delta R B_\varepsilon n_\varepsilon \left(3\,[\varkappa]^{-1/3} + [\varkappa]^{-1}\right) \tag{3.5.7}$$

for the total energy loss.

It is interesting that, in spite of the cursory character of the considerations, formula (3.5.7) gives, for $\varkappa < 1$, a dependence of $\overline{\Delta_\varepsilon E}$ on v and z_2 which corresponds approximately to the observations regarding α-rays for which the stopping power in heavy materials, over a considerable energy interval, is roughly proportional to v^{-1} and to a power of z_2 which does not differ much from $^1/_3$. Even if the numerical values of (3.5.7) cannot be expected to be very accurate, we should thus be justified in using the expression in comparing the stopping power for α-rays and fission fragments.

As regards the relative contribution to the stopping, of free collisions and resonance effects, we further notice that, while for $\varkappa < 1$ the two contributions are, of course, equal, the free collisions rapidly become dominating for $\varkappa > 1$ in an even more pronounced manner than in light substances. Thus, for $\varkappa = 8$, corresponding approximately to the case of fission fragments over a large part of the range (cf. § 5.3), the contribution of the resonance effects will amount to only about 15 %.

For $v \sim c$, modifications in the above formulae are necessary. Not only must relativistic corrections be included, but the procedure on which (3.5.7) is based is no longer justified, since, for very large values of u_s, we are outside the scope of the simple estimate (3.5.2). In this velocity region, the analysis of Bloch (1933a), referred to above, should be more adequate and it is of interest that essentially the same results may be obtained from simple relations, somewhat more general than (3.5.2), for the dependence of n on u_s and z_2. By such a procedure (A. Bohr 1948), it is also possible to estimate the influence of the atomic interaction (cf. p. 72) which, for very large values of γ, becomes determining for the stopping effect.

For the mean square deviation of $\Delta_\varepsilon E$, determining for the straggling, we get, instead of (3.4.5) holding for light materials,

$$\Omega_\varepsilon^2 \sim N \Delta R B_\varepsilon T_m n_\varepsilon [\varkappa]^{-1/3}, \qquad (3.5.8)$$

since $n_\varepsilon[\varkappa]^{-1/3}$ represents the approximate number of electrons which contribute essentially to the energy loss in free collisions, as follows from the considerations leading to (3.5.6).

An estimate of the ionization effects may, of course, be attempted on similar lines but, since, in contrast to the stopping and straggling, the ionization depends primarily on the most loosely bound electrons in the atoms, we are outside the proper region of applicability of the simple approximation procedure.

In all the estimates in this chapter, the particle velocity v has been assumed to be large compared with v_0. For lower velocities, however, where $v \gtrsim v_0$, an estimate of penetration effects presents great difficulties and, in particular, the phenomena become complicated due to the influence of the processes of electron capture and loss by the incident particle. These processes will, in fact, not only be determining for the effective charge of the particle in collisions with atomic electrons, but will for decreasing velocities in themselves constitute an essential source of energy transfer. To these questions we shall return in connection with the discussion of the capture and loss phenomena in Chapter 4.

CHAPTER 4.

Capture and Loss of Electrons in Atomic Collisions.

§ 4.1. Survey of the Problem.

The first evidence of capture of electrons by high speed particles in passing through matter was obtained by HENDERSON (1922), who observed that a beam of α-rays contains a fraction of singly charged particles which increases rapidly with decreasing velocity. This phenomenon was examined in greater detail by RUTHERFORD (1924), who showed that, over the whole range of the α-rays, continual capture and loss of electrons take place, and was even able to measure the frequency of the separate processes along the different parts of the range. Thus, the effective cross-section σ_l for electron loss in air was found to vary approximately as the inverse velocity, while the cross-section for capture σ_c varied roughly as v^{-6}. Moreover, the ratio between the probabilities for capture and loss was estimated to be approximately the same for all substances examined.

In contrast to electron loss which can be compared with a simple ionization process, electron capture is obviously a more complicated phenomenon involving the interaction of at least three particles. A theoretical treatment of this phenomenon was first attempted by FOWLER (1924), who compared the balance between capture and loss of electrons by α-rays with a thermodynamical equilibrium between He^{++} and He^{+} in an electron atmosphere of a temperature corresponding to an average electron velocity equal to v, and a density comparable with that of the electronic distribution in the atoms. Although such considerations were found to be in suggestive agreement with the experimental results, the comparison with a thermodynamical equilibrium cannot be upheld in detail, especially as regards the interaction between an α-particle and light atoms in which the orbital electron velocities are small compared with v. In fact, in this case, the velocities of the electrons relative to the α-particle will all have practically the same direction, in contrast to the requirements of a thermal distribution.

A closer examination of the mechanism of the capture process also shows that the influence of the mutual interaction of the electrons in the atom on the capture phenomenon will be only small compared with the effect of the momentum changes which take place in the nuclear field. Preliminary considerations of a qualitative character (Bohr 1925) indicated, in particular, that the balance between capture and loss should be essentially different in the lightest and in heavier substances. This was verified by Jacobsen (1926), who found that σ_c for high speed α-particles in hydrogen was vanishingly small compared with σ_c in air, while σ_l in hydrogen was of the same order of magnitude as in other substances and could be determined with considerable accuracy. A detailed theory of electron capture by α-rays in light and in heavy substances was developed by Thomas (1927) on the basis of classical mechanics and, after the development of quantum mechanics, the capture problem was reconsidered by Oppenheimer (1928) and especially by Brinkman and Kramers (1930). The difference between the results of the latter authors and of Thomas illustrates again, as we shall see, in a characteristic manner the scope of the different approximation methods in atomic mechanics.

In the following paragraphs, a general survey of electron capture and loss by penetrating particles will be attempted with special regard to the dependence of the phenomena on the charge and velocity of the particles. To this purpose, we shall first consider the case of α-rays in various stopping materials and, thereafter, discuss the problem of the balance between loss and capture by highly charged particles passing through matter. The latter problem is of particular interest in connection with the behaviour of fission fragments which, over the whole range, carry a considerable number of bound electrons which steadily increases as the fragment is slowed down. The effective charge in electronic collisions, therefore, varies essentially along the path with the result that the range velocity relation of fission fragments exhibits peculiar differences from that of lighter particles.

§ 4.2. Cross-Section for Electron Loss by Light Nuclei.

The problem of electron loss presents, as already mentioned, a close similarity to the ionization by high speed particles and may be treated on the lines of the considerations in Chapter 3. The analogy is particularly clear if we consider the system of reference in which the α-particle is at rest and subjected to the bombardment of the atoms in the stopping material. The problem is especially simple in hydrogen and helium, where the orbital dimensions of the atomic electrons are larger than or comparable with the radius a_α of the bound state of the electron round the α-particle. In fact, in this case we may, in close collisions where the atoms penetrate each other, consider the ionizing effects of the atomic particles approximately independent of each other. For velocities large compared with v_0 we have $\varkappa < 1$, and thus, according to the considerations in § 3.3, the collisions in question just constitute the so-called "free collisions" ($P < a_\alpha$ (cf. p. 86)) in which the binding forces may be disregarded. In more distant collisions ($P > a_\alpha$), however, where the atoms pass outside the region of the electron bound by the α-particle, the effects of the individual atomic particles in the neutral atom will to a large extent compensate each other. The "resonance effects" which, just in the case of $\varkappa < 1$, contribute the larger part of the ionization processes produced by a single charged particle will, therefore, here be of only minor importance.

With the notations of Chapter 3, we thus have to a first approximation

$$\sigma_l = \frac{1}{N \Delta R} \sum (\omega_I)_f, \tag{4.2.1}$$

where the sum refers to the effects of the nucleus and the electrons in the atom of the stopping material. From (3.4.9) and (3.1.2) we consequently get

$$\sigma_l = \frac{2 \pi \varepsilon^4}{\mu v^2} \left(z_2^2 + z_2\right) \frac{1}{I}, \tag{4.2.2}$$

where I is the energy required for the removal of the electron from the particle, and z_2 is the atomic number of the substance penetrated. Putting $I = \frac{1}{2} \mu z_1^2 v_0^2$, and introducing a_0^2 from (2.1.1) as a convenient measure of the cross-section, (4.2.2) may be written, by means of (2.1.5),

$$\sigma_l = 4\pi a_0^2 z_1^{-2} \left(z_2^2 + z_2\right) \left(\frac{v_0}{v}\right)^2. \tag{4.2.3}$$

In the case of α-particles ($z_1 = 2$) in hydrogen ($z_2 = 1$), we find

$$\sigma_l = 2\pi a_0^2 \left(\frac{v_0}{v}\right)^2, \tag{4.2.4}$$

an expression which agrees satisfactorily with the empirical data. Thus, JACOBSEN (1926) found for $v \sim 8 v_0$ a mean free path for loss of about $6 \cdot 10^{-3}$ cm at N.T.P., corresponding to $\sigma_l = 3 \cdot 10^{-18}$ cm², while (4.2.4) gives $\sigma_l = 2.5 \cdot 10^{-18}$ cm².

For stopping materials of higher atomic numbers, where for the most firmly bound electrons $a_s < a_\alpha$, these electrons and the nucleus will, even in close collisions, no longer act independently on the electron carried by the α-particle, but the total effect will more resemble that of a screened nuclear field, with the result that σ_l will be considerably smaller than given by (4.2.3). The inadequacy of this formula for very large values of z_2 is also evident from the circumstance that σ_l would become large compared with atomic dimensions. In the limiting case of large z_2, an estimate of σ_l is especially simple, since the field inside the atom will be so intense that almost any collision in which the α-particle penetrates the atomic region will, if only $v \gg v_0$, lead to the removal of the bound electron. In such cases we must, therefore, expect a value for σ_l of the same order as πa_0^2 and largely independent of z_2 as well as of v.

For intermediate values of z_2, a closer analysis of the effect of the atomic field is necessary. Here, we have a problem related to the questions of excessive screening, discussed in § 1.5 and also referred to in § 2.1 in connection with the scattering of fast electrons in nuclear collisions. In the present case, it follows from (2.1.9) that, for instance for air ($z_2 \sim 8$), we have $\zeta \sim 1$ for $v \sim 6\,v_0$. For such values of ζ, the deflection field may in the determining region be compared with a potential of the type (1.5.5) for $n = 2$, corresponding to a screening potential (1.4.1) in the region around $r = a = a_0 z_2^{-1/3}$. For the value of k we thus have, according to (1.5.6),

$$k = e^{-1} z_2^{2/3} \varepsilon^2 a_0. \tag{4.2.5}$$

As discussed in § 1.5, the deflection in such a field may be treated by means of classical mechanics as regards angles larger than the value (1.5.2) which may be written

$$\vartheta^* = z_2^{1/3} \frac{v_0}{v}. \tag{4.2.6}$$

For our purpose, it is of importance to compare this angle with the deflection ϑ_I corresponding to an energy transfer equal to the ionization energy $I = \frac{1}{2} \mu z_1^2 v_0^2$. From (1.1.8) we have, assuming $z_1 v_0 < v$,

$$\vartheta_I = z_1 \frac{v_0}{v}, \tag{4.2.7}$$

which shows that, for $z_1 \lesssim z_2^{1/3}$, we may approximately account for the ionization processes by means of classical mechanics.

For the cross-section for loss, we may thus write $\sigma_l \sim \pi i^2$, where i is the impact parameter corresponding to $\vartheta = \vartheta_I$ and find, by means of (1.5.7), (4.2.5), and (4.2.7),

$$\sigma_l \sim \pi a_0^2 \cdot z_2^{2/3} z_1^{-1} \left(\frac{v_0}{v}\right). \tag{4.2.8}$$

It is of particular interest that this expression varies like v^{-1} just as RUTHERFORD found for α-rays in air over a large part of the range. For $v \sim 8\,v_0$, RUTHERFORD obtained a mean free path for loss of 0.011 mm at N.T.P., corresponding to $\sigma_l = 1.6 \cdot 10^{-17}\,\mathrm{cm}^2$, while formula (4.2.8) gives $\sigma_l \sim 2 \cdot 10^{-17}\,\mathrm{cm}^2$. In view of the approximations involved in the calculations, the agreement may be considered to be satisfactory.

§ 4.3. Cross-Section for Electron Capture by α-Particles.

While electron loss is essentially a two-body problem, the electron capture presents us, as already stressed, with a phenomenon in which energy and momentum are exchanged between at least three particles. In a treatment of the problem on classical mechanics (THOMAS 1927), the electron capture in light substances was described as a double process; the first part being a collision between the incident particle and an atomic electron, in which the latter obtains a velocity of the magnitude v, and the second part being a collision of this electron with the atomic nucleus, resulting in a deflection after which the electron velocity also in direction coincides closely with that of the capturing particle. Since in each of these processes we have to do with large angle deflections, one might have expected that such a calculation would give essentially correct results, even if the quantity $\varkappa$ is small compared with unity, and classical pictures, therefore, are inadequate in analyzing the details of the collision.

It must be realized, however, that in the capture phenomena we have not simply to do with two separate collisions the individual effects of which, as in the problems discussed in Chapter 1, are defined by the wave-functions at large distances from the scattering centre. On the contrary, electron capture presents us with an intricate collision process for the result of which the interference of the scattered wavelets during the overlapping of the atomic fields may be decisive. In fact, as shown by BRINKMAN and KRAMERS (1930) in their detailed treatment of this phenomenon by means of BORN's approximation, the probability of capture is negligible, except in collisions where the two nuclei pass each other at distances comparable with the wave-length λ corresponding to an electron with velocity v. It is, therefore, not surprising that their calculation gives a dependence of σ_c on the charges of the nuclei and on their relative velocity which differs essentially from that obtained by means of classical mechanics.

For the cross-section for capture by a nucleus z_1 of an electron bound to a nucleus z_2, BRINKMAN and KRAMERS derived an expression which, for v large compared with the orbital velocities $z_1 v_0$ and $z_2 v_0$, may be approximately written

$$\sigma_c = \frac{2^{18}}{5} \pi a_0^2 z_1^5 z_2^5 \left(\frac{v_0}{v}\right)^{12}. \qquad (4.3.1)$$

For smaller velocities, the calculation cannot claim any considerable accuracy, since the approximation procedure is justified only if the quantities $\varkappa_1 = 2 z_1 \frac{v_0}{v}$ and $\varkappa_2 = 2 z_2 \frac{v_0}{v}$ are small. The very rapid variation with v implies that σ_c becomes extremely small for $v \gg v_0$ and, therefore, explains the negative results of JACOBSEN's attempt to measure the cross-section for high speed α-particles in hydrogen.

For particle velocities of the same order of magnitude as the orbital velocities of the electron before and after its capture, neither classical mechanics nor the Born approximation can yield accurate results. Still, for $\varkappa_1 \sim \varkappa_2 \sim 1$, the calculations of Thomas as well as those of Brinkman and Kramers give, as might be expected, values for the capture cross-section of the same order of magnitude as the orbital areas. In such cases, the cross-section for capture will be comparable with the cross-section for loss, and the particle will during a considerable fraction of its path carry a bound electron. This is in agreement with experiments on slow protons or α-particles in hydrogen or helium which, for velocities of the order of magnitude of v_0, have proved that the mean charge of the particles differs essentially from one and two units, respectively.

Recently, a discussion of these measurements has been given by Knipp and Teller (1941), who have attempted from the empirical values of the mean square of the particle charge to estimate the corrections which, on account of the capture phenomena, are to be introduced into the stopping formulae for slow α-particles or protons. In the application of such corrections it must, however, be remembered not only that the approximations used in the deduction of the stopping formulae are not valid for $v \sim v_0$, but also that the capture processes themselves under such circumstances constitute a considerable source of energy transfer.

For electron capture by α-particles in heavier substances, estimates of σ_c have been attempted by Thomas (1927) on classical mechanical calculations, and by Brinkman and Kramers (1930). Although both mothods gave approximate agreement with the experimental results, such more detailed calculations cannot claim great accuracy. In fact, the main contribution to the capture in heavy atoms containing electrons of widely different binding energies will be due to the electrons with orbital velocities comparable with v and, just in this case, the capture process can neither be followed in detail by means of classical pictures nor be rigorously treated by means of the Born approximation method. For the sake of the following discussion, it is instructive, however, to show how a qualitative estimate of σ_c may be obtained by simple statistical considerations.

To this purpose, we note that electrons with orbital velocities $u \sim v$, due to the action of the atomic field, will be subjected to a momentum change comparable with μv within a time interval corresponding to the passage of the particle through the orbital region. Accordingly, the velocity change which in lighter atoms demands a special double collision may here result from any collision in which an energy of the order $^1/_2\,\mu v^2$ is transferred to an electron of orbital velocity comparable with v. Now, quite independently of the value of $\varkappa$, the cross-section for such collisions will, according to (3.1.1) and (3.1.2), approximately be given by

$$\sigma \sim 4\,\pi a_0^2 z_1^2 \left(\frac{v_0}{v}\right)^4 . \qquad (4.3.2)$$

Since the dimensions of the atomic region occupied by the electron of orbital velocities $u \sim v$ will be small compared with the orbital radius $a_1 = a_0 z_1^{-1}$ of the electron after capture, the probability f that capture results from the first collision will, therefore, be of the same order of magnitude as the fraction of the velocity space corresponding to velocities relative to the incident particle comparable with $z_1 v_0$ or

$$f \sim z_1^3 \left(\frac{v_0}{v}\right)^3 . \qquad (4.3.3)$$

Further, for $v \gg v_0$, the number n of atomic electrons with orbital velocities u comparable with v $\left(\text{say, } \frac{1}{2}\, v < u < \frac{3}{2}\, v\right)$ is approximately, according to (3.5.2),

$$n = z_2^{1/3} \left(\frac{v}{v_0}\right) \tag{4.3.4}$$

and, consequently, we get

$$\sigma_c \sim \sigma f n \sim 4\,\pi\, a_0^2\, z_1^5\, z_2^{1/3} \left(\frac{v_0}{v}\right)^6 . \tag{4.3.5}$$

This simple estimate is in agreement with RUTHERFORD's results that σ_c for α-particles in air is nearly proportional to v^{-6}. Moreover, for α-rays with $v = 1.8 \cdot 10^9$ cm sec$^{-1} \sim 8\,v_0$, RUTHERFORD found a mean free path for capture in air at N.T.P. of 2.2 mm, corresponding to $\sigma_c = 8 \cdot 10^{-20}$ cm^2, while an estimate based on (4.3.5) gives $\sigma_c \sim 10^{-19}$ cm^2. The close agreement is, of course, more or less accidental, in view of the cursory character of the approximations. It is, furthermore, of interest to note that, compared with the estimates of σ_l in § 4.2, formula (4.3.5) gives a value for the ratio between σ_c and σ_l which varies only slowly with z_2 in heavier substances, in conformity with the observations.

§ 4.4. Balance between Electron Capture and Loss by Highly Charged Particles.

While, over the major part of the range, high speed protons or α-particles only seldom carry an electron, the situation is entirely different for heavy nuclei like fission fragments which, even at the beginning of their range, carry a large number of bound electrons. This difference is at once explained by the circumstance that such highly charged nuclei are able to bind electrons in states with orbital velocities u greater than even the initial values of the particle velocity v. In fact, in collisions with the atoms of the stopping material, the removal of electrons with $u \gg v$ is impossible or at any rate very improbable, while the less firmly bound electrons are readily removed in such collisions. As regards capture, the situation is reversed. Whereas, in atomic collisions, electrons are easily captured into states of orbital velocities $u \gtrsim v$, capture of electrons into states for which $u \ll v$ is a process which, generally, is very improbable compared with loss. Without any detailed estimate of the cross-section for capture and loss, we may, therefore, conclude that highly charged nuclei on the average carry a number of bound electrons approximately corresponding to the number of electrons in the neutral atom for which $u > v$.

In a preliminary discussion of the stopping problems for fission fragments (Bohr 1940 and 1941), it was shown that the general trend of the range velocity curves could be accounted for by an estimate of the effective charge number z_1^* based on such assumptions. Similar views were used by Lamb (1940) who identified z_1^* with the charge number of the ion formed by the removal of all electrons with binding energies less than $\frac{1}{2}\mu v^2$. A simple, comprehensive formula for z_1^* (Bohr 1941) is, according to (3.5.2), given by

$$z_1^* \sim z_1^{1/3} \frac{v}{v_0} \qquad (4.4.1)$$

applying to particle velocities in the region $v_0 < v < z_1^{2/3} v_0$. On the basis of a more detailed examination of the velocity distribution of the atomic electrons by means of the Thomas-Fermi method, Knipp and Teller (1941) and Brunings, Knipp and Teller (1941) have attempted a more accurate determination of z_1^*, involving a closer estimate, by means of semi-empirical methods, of the ratio between the particle velocity and the orbital velocity of the most loosely bound electrons in the ion. As the authors themselves emphasize, however, the very assumption of a critical orbital velocity, for which capture and loss balance, involves in itself a considerable element of arbitrariness.

The first direct measurement of the charge of fission fragments was obtained by Perfilov (1940) who, from the curvature of the paths in a magnetic field, estimated the initial charge numbers to be about 20. Since, for the two main groups of fission fragments from uranium, z_1 is about 38 and 54, and the initial velocity of the order of $6v_0$ and $4v_0$, respectively, Perfilov's estimate agrees approximately with (4.4.1). A more detailed study has recently been performed by Lassen (1945 and 1946), who was able to measure the charge of the two groups separately and also obtained information concerning the variation of z_1^* along part of the range. The initial charge numbers were found to be 20 and 22 for the light and the heavy group, respectively. These values are again of the order of magnitude of the estimate (4.4.1) although, from this simple formula, slightly higher values for the light than for the heavy group were to be expected (cf. Lassen 1945). The experimental results are, however, readily explained by the fact that, at the beginning of the range, the lighter fragments are stripped by more than half the electrons in the neutral atom and that, therefore, we are in a region where the effective quantum numbers of the atomic electrons can no longer be considered to have a constant value close to $z^{1/3}$ (cf. p. 101). This circumstance also explains the observation by Lassen that the variation with velocity of z_1^* for the light fragment at the beginning of the range is slower than corresponding to (4.4.1).

In Chapter 5, we shall discuss the range velocity relation for fission fragments in some detail and, in particular, investigate what information regarding the effective charge of the fragments may be derived from range and ionization measurements. As we shall see, the experimental results agree reasonably well with the dependence of z_1^* on z_1 and v to be expected from the simple arguments here indicated. Although this dependence is found to be closely the same for all stopping materials, there is still some indication that z_1^* for a given velocity is slightly higher in the lightest substances, a point which may be explained on much the same lines as the anomalous behaviour of α-particles in hydrogen as regards electron capture. To investigate this point it is necessary, however, to look into the problem of the balance between capture and loss somewhat more closely than in the approximate estimate of z_1^*.

As regards light substances, we may apply the considerations of § 4.2 and § 4.3. Thus, in analogy to (4.2.2) and (4.2.3), one finds

$$\sigma_l \sim 4\pi a_0^2 z_1^{1/3} z_2^2 \left(\frac{v_0}{v}\right)^3 \tag{4.4.2}$$

when taking into account that the removable electrons, numbering approximately $z_1^{1/3}\dfrac{v}{v_0}$, have ionization potentials of the order of $\dfrac{1}{2}\mu v^2$. In estimating σ_c, use may be made of the principle of detailed balance according to which the cross-section for a transfer of an electron from one ion to another, in specified quantum states, must be the same as the cross-section for the reverse process, as illustrated from the symmetry of the expression (4.3.1). It is true that the capture of an electron by a heavy ion from a light atom is not the direct reversal of the capture process by a light ion in collision with a heavy neutral atom. Still, the lack of outer electrons in the ion mainly implies an increase in the number of empty states into which the electron can be captured, and this effect is largely balanced by the number of electrons which can be captured in the converse process. As an approximate estimate, we may thus write

$$\sigma_c \sim 4\,\pi\, a_0^2\, z_1^{1/3}\, z_2^5 \left(\frac{v_0}{v}\right)^6, \tag{4.4.3}$$

corresponding to (4.3.5) by an interchange of z_1 and z_2.

In heavy stopping materials, the capture and loss processes are, of course, very difficult to follow in detail, but it is evident that there will be a considerable probability of an exchange of electrons between the ion and the stopping atom in any collision where the ion penetrates the atomic region containing electrons of orbital velocities comparable with v. As follows from considerations analogous to those implied in (3.5.1), the radial extension of the ion will be approximately given by

$$a_1^* \sim a_0 \frac{\nu_s^2}{z_1^*} \sim a_0\, z_1^{1/3} \frac{v_0}{v}, \tag{4.4.4}$$

where we have put $\nu_s \sim z_1^{1/3}$ and introduced z_1^* from (4.4.1). A similar expression will hold for the size of the atomic region active in the exchange processes. For σ_l and σ_c which, in the case considered, will be of the same order of magnitude, the symmetrical expression

$$\sigma_l \sim \sigma_c \sim \pi\, a_0^2 \left(z_1^{1/3} + z_2^{1/3}\right)^2 \left(\frac{v_0}{v}\right)^2 \tag{4.4.5}$$

therefore suggests itself as a rough estimate.

Comparing formula (4.4.5) with (4.4.2) and (4.4.3), we notice the far smaller dependence on z_2 and v of the cross-sections in heavy as compared with light materials. In particular will the ratio $\frac{\sigma_c}{\sigma_l}$ which, in the former case, is always of the order of unity, in the latter case decrease rapidly with v and will, especially in hydrogen ($z_2 = 1$), be very small for $v \gg v_0$. For such velocities we may, therefore, as already mentioned, expect the average charge of the ion to be somewhat larger in the lightest than in heavier materials, although it will, of course, always remain of the same order of magnitude as the estimate (4.4.1).

For the discussion in the next chapter, of the stopping effect of fission fragments, it is moreover essential to examine to what extent such heavy ions may be regarded as point charges. To this purpose, we note in the first place that, for a collision between a free electron and a high speed ion of charge number (4.4.1), we have from (3.2.3)

$$\varkappa = 2\, z_1^* \frac{v_0}{v} \sim 2\, z_1^{1/3} \qquad (4.4.6)$$

which, for fission fragments, is essentially larger than unity. The collisions can, therefore, with high approximation be treated on classical mechanics and, according to the considerations in Chapter 1, the collision diameter b, corresponding to encounters between the charges $z_1^* \varepsilon$ and $-\varepsilon$, will be a suitable measure for the minimum impact parameters contributing essentially to the stopping effects. By a comparison between (1.1.4) and (4.4.4) it is seen that $b \sim 2\, a_1^*$, and the internal structure of the ion should, therefore, have only a very small influence.

While a survey of the properties and behaviour of the fission fragments can be obtained by means of simple arguments, if v is large compared with v_0, the problem becomes far more intricate for velocities approaching v_0. Not only will all estimates of z_1^* become very uncertain when the particles tend to be neutral, but it will no longer be justified to regard the system as a point charge in electronic collisions. As we shall see, however, the great latitude in estimates of electronic stopping is rendered relatively unimportant by the preponderant influence of nuclear collisions on the stopping effect towards the end of the range of such heavy particles.

CHAPTER 5.

Range Velocity Relations.

§ 5.1. General Aspects of the Problem.

As is well known, range measurements often offer a convenient means of determining the initial velocity of high speed atomic particles. In fact, in many cases, the majority of the particles will be gradually stopped without suffering greater deflections and, therefore, a beam of uniform velocity reaches a fairly well defined distance from the source. Just in this respect, there is an essential difference between the behaviour of electrons and that of heavier particles. While, in the latter case, the accumulative effect of individual smaller collisions will produce a stopping phenomenon resembling that of a body moving through a viscous medium, we meet in the former case, due to the frequent large scattering, with a phenomenon more analogous to the absorption of a beam of electromagnetic radiation which during its propagation is attenuated according to an exponential law.

Such typical absorption effects are especially pronounced in the case of high speed electrons penetrating through heavy substances, where the large angle deflections will occur with great probability even within sections of the path in which the mean energy loss constitutes only a small fraction of the total kinetic energy. In fact, the stopping power will for fast β-rays be roughly proportional to the atomic number z_2 of the substance, while the probability of nuclear scattering within a given section of the range will be proportional to z_2^2. Thus, a simple estimate (BOHR 1915) shows that, for medium values of z_2, the probability of a beam of electrons being dispersed before the particles have lost the major part of their kinetic energy is of the order of unity, in agreement with experiments according to which, in the lightest substances, β-rays possess a rather well-defined range while, in heavy substances, the intensity of the beam falls off nearly exponentially. The problem has been considered more closely by BOTHE (1923) on the basis of an examination of the compound scattering.

The main results may be directly illustrated by means of the analysis given in Chapters 2 and 3. Thus, from formula (2.5.10) for the compound scattering, one gets, by introducing the most probable energy loss in electronic collisions, determining for the stopping effect,

$$(\Psi^*)^2 = z_2^2 \frac{L_\nu}{L_\varepsilon} \frac{\overline{\Delta_\varepsilon^* E}}{E}, \tag{5.1.1}$$

where L_ν and L_ε stand for the logarithmic terms in the nuclear and the electronic stopping formulae (2.4.6) and (3.4.7), respectively. Since L_ε is comparable with $z_2 L_\nu$, we see that, for high values of z_2, we may have $\Psi^* \sim 1$ even for $\overline{\Delta_\varepsilon^* E} \ll E$, while it follows from a closer evaluation of the logarithmic terms that, only in the lightest substances, Ψ^* will remain small for $\overline{\Delta_\varepsilon^* E} \sim E$.

In case of fast heavy particles, the scattering effects will, in general, be of only minor importance and we have, therefore, to do with a well-defined range, except for a certain straggling due to the statistical fluctuations in the accumulative stopping effects. Assuming that $v \ll c$, we get from the formulae in Chapters 2 and 3, with a similar notation as in (5.1.1),

$$\frac{dv}{dR} = 2\pi N \frac{(z_1^*)^2 \varepsilon^4}{m_1 \mu v^3} L_\varepsilon + 2\pi N \frac{z_1^2 z_2^2 \varepsilon^4}{m_1 m_2 v^3} L_\nu, \qquad (5.1.2)$$

where z_1^* in the first term represents the effective charge number of the particle in electronic collisions which, especially for highly charged particles, may be essentially smaller than z_1 on account of electron capture.

Still, due to the appearance of μ in the denominator, the first term in (5.1.2) will in general be large compared with the second for velocities exceeding v_0. For smaller velocities, the simple theory of electronic stopping effects ceases to be valid, but, due to the fact that in this region nuclear stopping plays the essential part especially for large values of z_1, we may often to a first approximation neglect the first term entirely. The value of v for which nuclear stopping begins to be decisive will, however, due to the presence of the factor z_1^2 in the second term in (5.1.2), be considerably higher for particles like fission fragments than for protons or α-rays.

While for α-rays the influence of nuclear collisions, which is manifested in the irregularity of cloud- chamber tracks in the immediate neighbourhood of their end, is only of little significance in range determinations, nuclear stopping will, in the case of fission fragments, be of importance over quite an appreciable fraction of the range, as appears from the marked curvature of the tracks (cf. p. 62). For the closer treatment of the range problem it is essential that the second term in (5.1.2) retains its validity down to velocities considerably lower than v_0. Still, as discussed in § 2.3, an expression of the type (2.3.9) must be applied in the case of very slow particles.

The main purpose of the following discussion will be to examine the characteristics of the range velocity relations for particles of different charge and velocity. In § 5.2, we shall briefly review the situation for particles like protons or α-rays, where capture phenomena are of only minor importance. In § 5.3, the problem of highly charged particles will be considered, with special reference to the properties of fission fragments. Finally, in § 5.4 we shall briefly discuss the stopping problems for particles of smaller initial velocity like recoil atoms in nuclear disintegrations.

§ 5.2. Range Relations for High Speed Light Nuclei.

Due to its practical importance in nuclear researches, the problem of range velocity relations for particles like protons and α-rays has been much discussed and has especially been thoroughly treated by LIVINGSTON and BETHE (1937) who, on the basis of BETHE's theory, have developed semi-empirical formulae for the dependence of the stopping power on velocity for different substances. We shall, therefore, confine ourselves to a survey of some of the more principal results by means of the considerations in Chapters 2 and 3.

For the major part of the range, we can here put $z_1^* = z_1$ and also neglect nuclear stopping effects. The decisive point is, thus, the dependence of L_ε on z_2 and v. In very light substances and for $v \gg v_0$, we have approximately L_ε proportional to z_2 and only slowly varying with v. Putting $L_\varepsilon = z_2 L$, we get from (5.1.2), by introducing the quantities a_0 and v_0 from (2.1.1) and (2.1.5),

$$\frac{dv}{dR} = 2\pi N a_0^2 \frac{\mu}{m_1} z_1^2 z_2 \frac{v_0^4}{v^3} L. \tag{5.2.1}$$

For hydrogen, (3.4.3) gives, since $\varkappa \gtrsim 1$, the values $L = 12.0$ and 9.2 for $v = 10\,v_0$ and $5\,v_0$, while the empirical values are $L = 11.7$ and 8.9, respectively. The small difference which is insignificant to our purpose is due to the circumstance that the value of u_s determining for η_s in (3.4.3) has to be chosen slightly greater than v_0, as follows from the detailed calculations of Bethe. From (5.2.1), the range may be obtained by simple integration, giving

$$R = \frac{1}{8\pi N a_0^2} \frac{m_1}{\mu} \frac{1}{z_1^2 z_2} \left(\frac{v}{v_0}\right)^4 \frac{1}{\bar{L}}, \tag{5.2.2}$$

where $\bar{L}$ is a suitable mean value of L which, for initial velocities large compared with v_0, is only little smaller than the value of $\bar{L}$ at the beginning of the range. Thus, for $v = 10\,v_0$, the empirical value to be introduced into (5.2.2) for hydrogen is $L = 9.6$.

In heavier substances containing electrons with orbital velocities greater than v, we find, according to (3.5.5) and (3.5.7), since $\varkappa \gtrsim 1$, a value of L_ε proportional to v. Putting $L_\varepsilon = f\frac{v}{v_0}$, we get from (5.1.2) the range velocity relation

$$R = \frac{1}{6\pi N a_0^2} \frac{m_1}{\mu} \frac{1}{z_1^2} \left(\frac{v}{v_0}\right)^3 \frac{1}{f} \tag{5.2.3}$$

which, as regards dependence on v, accounts for the empirical rule of Geiger. The value of f to be used should, according to the considerations in § 3.5, for large values of z_2, approach $8\,z_2^{1/3}$. For xenon this would give $f \sim 30$, which is close to the empirical value.

Also the phenomenon of range straggling is, for particles like α-rays, practically uninfluenced by nuclear collisions, but here it is necessary to consider somewhat more closely the statistical problems involved. In fact, a comparison of the expressions (2.3.7) and (3.4.5) for the mean square deviation of the energy losses over a given section of the range shows that Ω_ν^2 and Ω_ε^2 are of the same order of magnitude. While, however, the distribution of $\Delta_\varepsilon E$ is of a Gaussian type for any not too small part of the range, the distribution of $\Delta_\nu E$ will, in general, as follows from the considerations in § 2.4, be of an essentially different type, involving major fluctuations in the energy losses only for a small fraction of the particles.

As regards $\Delta_\varepsilon E$, we have a statistical distribution of the type (2.4.1) for which we may conveniently write

$$\Omega^2 (\Delta_\varepsilon E) = P_\varepsilon \Delta R. \tag{5.2.4}$$

In order to estimate the corresponding range straggling, we may proceed in the following manner (cf. Bohr 1915). The fluctuation in $\Delta_\varepsilon E$ will give rise to a Gaussian distribution of the values of ΔR corresponding to a fixed amount of energy loss ΔE with a mean square deviation

$$\Omega_\varepsilon^2 (\Delta R) = \Omega^2 (\Delta_\varepsilon E)\left(\frac{\Delta R}{\Delta E}\right)^2 = P_\varepsilon \Delta E \left(\frac{\Delta E}{\Delta R}\right)^{-3} \tag{5.2.5}$$

and, for the resulting straggling in total range, we have thus

$$\Omega_\varepsilon^2 (R) = \int_0^E P_\varepsilon \left(\frac{dE}{dR}\right)^{-3} dE. \tag{5.2.6}$$

Since P_ε for light stopping materials, according to (3.4.5), is independent of v and, for heavier substances, according to (3.5.8), is roughly proportional to v, it follows from (5.2.6) that, for range velocity relations like (5.2.2) or (5.2.3), by far the greatest contribution to the straggling is due to the high velocity part of the range.

Evaluating the integral we get, with the above notation,

$$\frac{\Omega_\varepsilon^2(R)}{R^2} = 4\frac{\mu}{m_1}\frac{1}{L} \tag{5.2.7}$$

in the case of light substances where the value of L corresponds with high approximation to the initial velocity. For α-particles in hydrogen, we shall thus expect a relative range straggling $\frac{\Omega_\varepsilon(R)}{R}$ of a little less than 1 %. For heavy substances, we get by a similar calculation, making use of the comprehensive formulae in § 3.5, holding for not too fast particles,

$$\frac{\Omega_\varepsilon^2(R)}{R^2} = \frac{3}{4}\frac{\mu}{m_1}, \tag{5.2.8}$$

which corresponds to a relative range straggling for α-rays of about 1 %, independent of stopping material and initial velocity.

In order to estimate the contribution of nuclear collisions to the straggling, we may proceed in a way quite analogous to the considerations in § 2.4 of the distribution of the energy losses $\Delta_\nu E$. Let us, thus, by δR^* denote a variation in the range due to a single collision and chosen in such a way that the mean number of collisions during the whole range giving rise to a loss of range larger than δR^* is just equal to one. The corresponding energy loss T^* will, of course, be different for the different parts of the range and will be given by

$$T^* = \delta R^* \left(\frac{dE}{dR}\right). \tag{5.2.9}$$

From the definition of δR^* we have, from (2.2.5), since $T^* \langle\langle T_m$ over practically the whole range,

$$1 = \int_0^R NB_\nu \frac{1}{T^*} dR = \frac{1}{\delta R^*} \int_0^E NB_\nu \left(\frac{dE}{dR}\right)^{-2} dE. \tag{5.2.10}$$

For the fluctuations in range due to individual energy losses smaller than T^*, which give rise to an approximate Gaussian distribution, we have from an expression analogous to (5.2.6), by introducing $P_\nu^* = NB_\nu T^*$ (cf. (2.4.7) and (5.2.4)),

$$\Omega_\nu^*(R) = \delta R^* = \int_0^E NB_\nu \left(\frac{dE}{dR}\right)^{-2} dE, \tag{5.2.11}$$

where we have made use of (5.2.9) and (5.2.10).

Using the range velocity relations for fast α-particles, we find from (5.2.11), by means of (2.2.3),

$$\frac{\Omega_\nu^*(R)}{R} = \frac{z_2}{L} \frac{\mu}{m_2} \tag{5.2.12}$$

in the case of light stopping materials and

$$\frac{\Omega_\nu^*(R)}{R} = \frac{3}{16} \frac{\mu}{m_2} z_2^{5/3} \left(\frac{v_0}{v}\right) \tag{5.2.13}$$

for heavy substances. From a comparison with (5.2.7) and (5.2.8) it is thus seen that $\Omega_\nu^*(R)$ amounts to only a few per cent of $\Omega_\varepsilon(R)$ even for the highest values of z_2. Looking apart from the rare cases in which an α-particle suffers a collision with a nucleus so violent that it loses a considerable part of its energy, the contribution of nuclear collisions to the range straggling for fast α-particles may, therefore, be entirely neglected.

Experimental investigations of the straggling of fast protons or α-rays meet in general with difficulties in securing sufficiently well defined conditions, and the values obtained are, therefore, likely to be too high. Cloud-chamber studies by Rayton and Wilkins (1937) of the straggling of fast α-rays have, however, given values only slightly greater than was to be expected from (5.2.7) and (5.2.8). More recently Bøggild (1948), by a study of cloud-chamber tracks of protons emitted in nuclear reactions, has obtained results which agree closely with the theoretical formulae. A further interesting test of the straggling theory is provided by the measurements of the range of mesons in photographic emulsions (Lattes, Occhialini and Powell 1947) which, for particles with a mass of about 200 μ and energies of about 4 MeV, give a relative range straggling of about 4 %. Since, in this case, L will be about 18 for hydrogen, formula (5.2.7) gives $\frac{\Omega}{R} = 3.3\ \%$, while, for heavier substances, formula (5.2.8) gives about 6 %. Considering that the emulsion consists of a mixture of light and heavy materials, the agreement must be regarded as satisfactory.

In the above estimates of range and straggling of light nuclear particles, it is assumed that electronic collisions are determining for the stopping over practically the whole range. While this is the case for particles with $v \gg v_0$, the situation becomes, of course, essentially different if the initial velocity approaches v_0. Under such circumstances, the very last part of the range, where nuclear collisions become effective, may constitute a not inconsiderable part of the whole range and may, in particular, be of significance for the range straggling. In fact, the expression for the relative straggling over this last part of the range will, instead of the ratio between the masses of electron and atomic nuclei, contain only ratios between nuclear masses and may, therefore, be of the order of unity. An indication that the relative straggling begins to increase considerably for velocities a few times larger than v_0 was obtained by Bøggild (1948) who, by measurements of cloud-chamber tracks of light nuclear fragments emitted in slow neutron reactions, found a straggling essentially greater than was to be expected for the effect of electronic collisions according to formulae like (5.2.7) or (5.2.8).

§ 5.3. Range Relations for Fast Heavy Ions.

The penetration phenomena for highly charged particles like fission fragments differ in several respects markedly from those exhibited by protons and α-rays. Thus, the measurements of the velocity of fission fragments along the range by means of branch statistics, as discussed in § 2.2, show (Bøggild, Broström and Lauritsen 1940) that, over the initial part of the range, the rate of velocity loss is approximately constant in contrast to the range velocity relations for α-rays, referred to in § 5.2, which are characterized by a rapid increase of dv/dR with decreasing velocity. Also measurements of the ionization of fission fragments along the path (Jentschke and Prankl 1939 and, especially, Lassen 1946a and 1948) show a fall in ionization over the initial part of the range which contrasts markedly with the steep rise in the ionizing power of α-rays with decreasing velocity.

A theoretical discussion of the penetration of fission fragments has been given by several authors (Beck and Havas (1939), Bohr (1940 and 1941), Lamb (1940 and 1941), Knipp and Teller (1941), and Brünings, Knipp and Teller (1941)). The characteristic difference in the range velocity curves for fission fragments and α-rays is, in the first instance, determined by the influence of electron capture on the effective charge z_1^*. While, for the end part of the range, nuclear collisions are to a large extent responsible for the stopping effect, the second term in (5.1.2) is, in the initial part of the range, negligible compared with the first and the variation of z_1^* along this part of the range may, therefore, be directly estimated from the measurements of dv/dR. In heavier substances where L_ε is roughly proportional to v, it follows thus from the approximate constancy of dv/dR that z_1^* is closely proportional to v, in accordance with the simple estimate (4.4.1). For a more accurate determination of z_1^* it is, however, essential to consider the dependence of L_ε on the quantity $\varkappa$ which, in contrast to the case of α-rays where $\varkappa \lesssim 1$, is here large compared with unity, as follows from the expression (4.4.6) which gives $\varkappa \sim 8$, approximately independently of velocity, if only $v \gg v_0$.

By means of the expressions for L_ε contained in (3.4.3) and (3.5.7) referring to light and heavy substances, respectively, and by using the ionization measurements of Lassen, one finds initial values of z_1^* which for all stopping materials are nearly equal and which, within the uncertainty of the experimental and theoretical estimates, coincide with the direct determinations (cf. § 4.4) of the total charge of the fragments. In accordance with these experiments, one also finds a variation of z_1^* with v which, for the heavy fragment group, is closely linear while, for the lighter group, it varies somewhat less rapidly with v in the high velocity region. This latter feature is clearly evident from the ionization curve obtained by LASSEN which, for the lighter fragments in contrast to the heavier fragments, for decreasing velocities exhibits an increasing slope until it gradually takes a more linear run.

As mentioned in § 5.1, an estimate of the total range as a function of the initial velocity is a more complicated problem for fission fragments than for α-rays due to the fact that the part of the range corresponding to $v \gtrsim v_0$ in the former case, in contrast to the latter, constitutes a considerable part of the whole range. On account of the approximate linearity of the range velocity curve for $v \gtrdot v_0$ it is, however, convenient to introduce the quantity R_{ex} defined by a simple extrapolation of this part of the curve to $v = 0$.

An expression for R_{ex} may be obtained from (5.1.2) by introducing the estimates for z_1^* and L_ε discussed in preceding chapters. Thus, in the approximation in which we can apply (4.4.1) for z_1^* and (3.5.7) for L_ε, we get, for the range of fragments with mass number A_1 and with initial charge z_1^*, the simple expression

$$\frac{R_{ex}}{R_\alpha} = 3 \frac{A_1}{(z_1^*)^2} g_\varkappa \tag{5.3.1}$$

where, in order to eliminate as far as possible the latitude in the estimate of L_ε, we have compared R_{ex} with the range R_α of an α-particle ($\varkappa \lesssim 1$) with the same initial velocity. In (5.3.1) it is assumed that $\varkappa$, as indicated by (4.4.6), is constant along the range and that L_ε, in accordance with (3.5.5), is proportional to v. The value of the quantity

$g_\varkappa$, representing the reduction in L_ε for fission fragments as compared with α-rays, is, for $\varkappa \sim 8$, close to 2.5.

The approximation involved in (5.3.1) should be well satisfied for the heavy fragment group in substances of not too low atomic number, where it is also found to be in close agreement with experimental data. In the case of the light fragment group where z_1^* varies somewhat more slowly than corresponding to (4.4.1) in the first part of the range, we must expect R_{ex} to be somewhat smaller than would correspond to (5.3.1), as is also confirmed by the experiments. In the case of stopping material of low atomic number, the ratio of R_{ex} to R_α must be expected in general to depend somewhat differently on the initial velocity since, in contrast to the case of high speed α-rays where L_ε, according to (3.4.3), varies only slowly with v, the large values of $\varkappa$ and the relatively small values of v for fission fragments imply that the logarithms vary more rapidly with velocity. By introducing in (5.1.2) the value (3.4.3) for L_ε one obtains, however, by integration, values for R_{ex} in approximate agreement with experimental data (cf. Lassen 1948).

The features connected with the appearance of $\varkappa$ in the expressions for L_ε also account, to a large extent, for the observed variations of the stopping power of fission fragments relative to α-rays in different materials. Thus, assuming that z_1^* for given z_1 and v, to a first approximation, is the same in all substances, we shall from formulae (3.4.3) and (3.5.7) for fission fragments with $v \sim 6 v_0$ (light group) expect a ratio between the stopping power of a *He* and a *H* atom of about 1.5, and between an *A* and a *He* atom of about 4.2, while for α-rays of similar velocity the corresponding ratios are 1.7 and 5.2, respectively. This agrees approximately with the measurements of Lassen (1948) who finds that, for the lighter group of fission fragments, in the first part of the range the ratios in question are 1.35 and 4.2, respectively. While the latter value coincides rather closely with the theoretical estimate, the ratio between the stopping powers of *He* and *H* is somewhat smaller than was to be expected if z_1^* is equal for the two substances. Although the evidence regarding this point is uncertain, it may be an indication that the number of captured electrons is slightly smaller in *H* than in other substances, as might also be expected from the considerations in § 4.4.

When the fission fragments have been slowed down to velocities comparable with or smaller than v_0, the estimate (4.4.1) for z_1^* is no longer valid and the effective charge in electronic collisions will decrease still more rapidly. As a consequence, the stopping effect would become very small and, due to the large energy still possessed by the fragments, a long tail in the range velocity curve would result, if the increasing influence of nuclear collisions did not counteract such a course. Looking, as a first approximation, apart from electronic collisions for $v < v_0$, the velocity loss in this region will be determined by the second term in (5.1.2) which holds down to velocities considerably below v_0. By integration, one obtains for the residual range R_0, corresponding to a velocity v,

$$R_0 = \frac{1}{8\pi N} \frac{m_1 m_2}{z_1^2 z_2^2 \varepsilon^4} v^4 \frac{1}{\overline{L_\nu}}, \qquad (5.3.2)$$

where $\overline{L_\nu}$ is a mean value of the logarithmic term which is close to the value of L_ν, corresponding to the velocity v.

It should be noted that formula (5.3.2) does not hold for $v \ll v_0$ since, as mentioned in § 2.3, it is a condition for the validity of the last term in (5.1.2) that the quantity ζ is small compared with unity. For fission fragments, the expression (2.1.7) gives $\zeta \sim {}^1/_{10}$ for $v = v_0$ and $\zeta \sim 1$ for $v = {}^1/_3\, v_0$, approximately independently of the stopping material. For still smaller velocities, the nuclear stopping will be given by (2.3.9) but, provided $v \sim v_0$, the resulting corrections to R_0 will be only negligible. For particles of smaller initial velocities, however, we get, as will be more closely discussed in the next paragraph, a range velocity relation of essentially different type.

In contrast to the range formulae referring to electronic collisions, formula (5.3.2) depends explicitly on m_2, and the relative importance of nuclear collisions may, therefore, be estimated directly from a comparison between the ranges of fission fragments in hydrogen and deuterium. According to the measurements of Bøggild, Arrøe and Sigurgeirsson (1947), the range in D is, in fact, larger than that in H by about 7 mm at N.T.P. Now, from the expression (5.3.2) for R_0, we find that this difference between D and H corresponds closely to $v = v_0$. To a first approximation we may, therefore, assume that, for velocities larger and smaller than v_0, the main stopping effect is due to electronic and nuclear collisions, respectively. This result also fits in quite well with the measurements of total ranges in various substances, which are approximately accounted for (cf. Lassen 1948) by the sum of R_{ex} and R_0, if v in (5.3.2) is put equal to v_0.

In order to estimate the range straggling for fission fragments, we may use similar considerations as in § 5.2. As regards the contribution from electronic collisions, we thus obtain from (5.2.6), by means of (5.3.1), expressions for $\Omega_\varepsilon(R_{ex})/(R_{ex})$ quite analogous to (5.2.7) and (5.2.8), except for slightly altered numerical constants arising from different range velocity relations. Due to the large mass m_1 of the fission fragments we, therefore, get values for the relative straggling of the order of 0.1 $^0/_0$ which, as we shall see, is negligible in comparison with the contribution of nuclear collisions to the straggling.

In estimating the latter contribution, it is important, like in § 5.2, to consider first the problem of the type of statistical distribution law governing $\Delta_\nu E$. According to the considerations in § 2.4, it follows that, in light substances where $m_1 \gg m_2$, we have to do with a simple Gaussian distribution of $\Delta_\nu E$, at any rate in that part of the range where nuclear collisions have any considerable influence on the stopping. In such cases, we get, in analogy to (5.2.6), since P_ν given by (2.3.7) is independent of v,

$$\Omega_\nu^2(R) = P_\nu \frac{1}{m_1^2} \int_0^v \left(\frac{dv}{dR}\right)^{-3} \frac{1}{v^2}\, dv \,. \qquad (5.3.3)$$

Now, for velocities large compared with v_0, where dv/dR for fission fragments is approximately constant, the integrant varies as v^{-2} while, for $v < v_0$ where dv/dR is roughly proportional to v^{-3}, the integrant contains a factor v^7. It follows, therefore, that the straggling essentially depends on the velocity region where $v \sim v_0$. In order to obtain a simple estimate of $\Omega_\nu^2(R)$, we may divide the integral into two parts corresponding to velocities larger and smaller than v_0, respectively, and replace dv/dR in the first part by $(dv/dR)_\varepsilon$ and in the second part by $(dv/dR)_\nu$, given by the first and second terms in (5.1.2), respectively. By an evaluation of the integral, one finds that the second term is preponderant and, in analogy to (5.2.7), one gets, by introducing the value of R_0 given by (5.3.2) for $v = v_0$,

$$\frac{\Omega_\nu^2(R)}{R_0^2} = 4\,\frac{m_2}{m_1}\,\frac{1}{L_\nu}, \tag{5.3.4}$$

where L_ν represents the logarithmic term in the nuclear stopping formula for $v \sim v_0$. For hydrogen, this formula corresponds to a relative straggling in the last part of the range of about 10 $^0/_0$, equivalent to some three per cent of the total range, of which R_0 constitutes about a third. For heavier substances still greater straggling is to be expected[1] but, when m_2 becomes comparable with m_1, special considerations must be applied since the statistical distribution of the ranges will no longer be of a simple Gaussian type.

§ 5.4. Range Relations for Atoms of Small Initial Velocity.

For particles with velocities small compared with v_0, we shall expect essentially different penetration phenomena than for high speed particles. Except for the case of slow electrons where peculiar quantum-mechanical resonance effects in atomic fields may occur, we have in this velocity region essentially to do with nuclear collision processes

[1] Note added in proof: Recently, S. Katcoff, J. A. Miskel and C. W. Stanley (Phys. Rev. **74**, 631, 1948) have succeeded, by means of radioactive chemical analysis, in studying separately the stopping in air of individual isotopes occurring among the fission fragments. The ranges are found to vary with charge, mass, and initial energy in approximate agreement with the simple theory outlined above. Moreover, the straggling, which in earlier investigations could not be separated from the variation in range of the different isotopes within each of the two easily distinguishable groups, was found to be about 5 $^0/_0$ for all the fragments, a result which also fits in with the theoretical expectations.

which to a large extent can be accounted for by means of simple mechanical considerations (cf. § 2.1). Since the quantity ζ will be comparable with or larger than unity, we meet with a practically uniform scattering in relative coordinates and, as already discussed in § 2.4 and § 2.5, we shall, therefore, expect typical diffusion effects, unless $m_1 \gg m_2$. Only in such cases, of which we have characteristic examples in the stopping, in not too heavy substances, of recoil atoms from radioactive disintegrations, we shall thus have to do with a well defined range.

In natural α-decay, the recoil atoms with $z_1 \sim 90$ will have initial velocities of about $^1/_5\, v_0$, corresponding to $\zeta \sim 5$, practically independently of z_2. Here, we have a typical case of excessive screening and, as discussed in § 1.5, the effective cross-section σ will, for $\zeta \gtrsim 1$, be of the same order as πa^2. Moreover, for not too large values of ζ, the effective part of the field will be roughly of the inverse cube type with a value of σ approximately given by (1.5.11). By means of (2.1.2) and (2.1.7), the stopping formula (2.3.9) thus gives, for $m_1 \gg m_2$,

$$\frac{dE}{dR} \sim \pi N a_0 z_1^{2/3} z_2 \varepsilon^2, \tag{5.4.1}$$

leading to a range velocity relation

$$R \sim \frac{1}{2\pi N a_0^2} \frac{1}{z_1^{2/3} z_2} \frac{m_1}{\mu} \left(\frac{v}{v_0}\right)^2, \tag{5.4.2}$$

where we have introduced a_0 and v_0 from (2.1.1) and (2.1.5).

For recoil particles with $z_1 \sim 90$ and corresponding to α-energies of 6 MeV, expression (5.4.2) gives a range in hydrogen at N.T.P. of about 0.4 mm, in satisfactory agreement with the measurements which are somewhat conflicting but, on the average, give a range of about $^1/_2$ mm. In water vapour at N.T.P., Joliot (1934) finds a range of about 0.08 mm, indicating that the stopping power of oxygen should be about 8 times larger than that of hydrogen, in accordance with formula (5.4.1).

As regards the range straggling, we get from (2.3.10) and (5.2.6), since, for $m_1 \gg m_2$, we have an approximately Gaussian distribution of the energy losses,

$$\frac{\Omega^2(R)}{R^2} = \frac{4}{3}\frac{m_2}{m_1}, \tag{5.4.3}$$

giving values for the relative range straggling in hydrogen of about 10 $^0/_0$. In oxygen, the expected value should be about 25 $^0/_0$, which is of the same order of magnitude as the straggling observed by Joliot in water vapour.

For still smaller velocities where ζ becomes very large compared with unity, formula (5.4.2) will no longer hold and, as mentioned in § 2.3, the effective cross-section will tend to coincide with the gas-kinetic cross-section, corresponding to a rate of energy loss proportional to the energy of the penetrating particle. For α-recoils, this circumstance will only affect the velocity range relation at the extreme end of the range, but in β-recoil, where the initial velocity is of the order of $\frac{1}{1000} v_0$, it will apply to the entire range. This problem has been more closely investigated by Jacobsen (1928), who has also taken the effect of the thermal velocities of the gas atoms into consideration when estimating the stopping of the recoil atoms. Experiments on these phenomena are very difficult, but are, as shown by Jacob-

SEN, in agreement with theoretical expectations within the uncertainty of the measurements.

Evidence of the characteristic difference in the stopping mechanism for particles with velocities large and small compared with v_0 is also afforded by ionization measurements. While, as mentioned in § 3.4, for high speed particles where electronic collisions are determining for the stopping, the energy expenditure per ion is largely independent of charge and velocity of the ionizing particle, the situation must be expected to be essentially different if a considerable part of or even practically the whole of the energy is transferred directly to the atomic nuclei, as is the case for the end part of the range of fission fragments and for the whole range of the recoil atoms. Also here, the experiments show that the energy loss is accompanied by an intense ionization which may be due to the fact that neither the primary collisions between the electron systems nor the secondary collision processes between the struck atoms and other atoms in the stopping material are strictly adiabatic. It is significant, however, that measurements of the energy expenditure per ion for α-recoil atoms (L. WERTENSTEIN 1913 and, especially, B. MADSEN 1945) give values which are several times larger than for α-rays and which increase rapidly with decreasing energy. For velocities of about $^1/_{20}\, v_0$, the curve obtained by MADSEN for the dependence on initial energy of the total number of ions produced by the recoil particle indicates an almost vanishing ionization corresponding to atomic collisions in which the electronic structures react in a practically adiabatic manner.

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